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CASSELL'S EDITION OF EUCLID.

THE

ELEMENTS OF GEOMETRY;

OR,

THE FIRST SIX BOOKS, WITH THE ELEVENTH AND TWELFTH,

OF

EUCLID.

FROM THE TEXT OF ROBERT SIMSON, M.D.,

Emeritus Professor of Mathematics in the University of Glasgow.

WITH CORRECTIONS, ANNOTATIONS, AND EXERCISES,

BY ROBERT WALLACE, A.M.,

OF THE SAME UNIVERSITY,

AND COLLEGIATE TUTOR OF THE UNIVERSITY OF LONDON.

BIBLIOTHECA BODLEIANA

LONDON:
W. KENT & CO. 51 & 52, PATERNOSTER ROW.

1855.

PREFACE.

THIS edition of Euclid's Elements of Geometry differs from the common editions with Dr. Simson's name attached to them, in several important particulars. *First*, the style has been simplified and modernised as much as possible, by removing much of its technicality; and in places where this was necessarily retained, numerous explanations have been added, especially in the Definitions. *Secondly*, many new Demonstrations of the Propositions have been given, *in addition* to those of Euclid, in order to bring the subject within the comprehension of different capacities. In not a few, while the spirit of the demonstration has been preserved, the original verbosity of the Greek, which was often retained by Dr. Simson in his translation, has been greatly curtailed; and in others, it has been altogether replaced by a new and better demonstration. *Thirdly*, to almost the whole of the propositions, there have been added, especially in the First Six Books, new Corollaries, Exercises, and Annotations of various kinds, tending to render the additions a species of short and running commentary on the immortal work of Euclid.

Explanations of all difficult terms in the science of Geometry have been given wherever they occur; and a style of punctuation in the different sentences of a proposition, and especially in the demonstration, has been adopted, which, it is believed, will be found of the greatest advantage to the student. This advantage will be discovered by comparison with other editions, and its utility will be seen from the following consideration. In reading the Demonstrations, the student is obliged to pause at every step, in order to make himself sure of the reasoning before he advances to the next step. This assurance may at once be supplied by his recollection of a previous proposition, a definition, a postulate, or an axiom; but if not, the reference is generally given, not at the margin, as in the common editions, but in the body of the text, just at the place where it is wanted. In either case, time is required to bring the reference vividly to the mind, and to assure it of the accuracy of the reasoning. In general, the time of a full period is not too much to enable the student to bring the memory to the aid of his judgment; and where the memory fails, to refer at once to the places in the book actually cited in the course of the argument. Every new period, therefore, has always the advantage of indicating a new step in the argument, and keeping the student awake to its progress.

The necessity of some method of this kind, to keep the mind alive to the march of the argument, was felt by the editor of one of our best recent editions of Euclid, Mr. Potts, of Cambridge. The plan adopted by him was to print every sentence or half sentence that contained a new step in the reasoning, *in a separate line;* thus giving the book the appearance of a book of limping poetry, and unnecessarily extending the letter-press over a much greater space; whereas, the great object of a reasoner should be to bring the words of the argument as closely together as possible, in order that the eye may help the mind. It was on this ground that algebraic signs were lately introduced into some editions of Euclid, and that a variety of other signs were employed to denote words and names of figures of frequent repetition. It was found, however, that by this plan nothing was gained, for the memory was now partially burdened by a new sort of language, a species of shorthand, more troublesome to write and read than the ordinary language of the science. Hence "Symbolical Euclids" have gradually and wisely given place to those editions in which the ordinary style of language is used, as much improved and abridged as the nature of the reasoning and the consistency of the argument will allow.

The above-mentioned simplifications, amendments, and additions, and other improvements which will be found on perusal, have been introduced into this edition, because it is chiefly intended for the use of the People. Euclid is now placed within the reach of all who are desirous of making themselves acquainted with this masterpiece of reasoning, with the foundation of all the sciences, with the basis of all the arts of design and machinery, and with the origin of all the processes relating to the measurement and calculation of surfaces and solids, required both in the arts of life and in the arts of production. Our space will not permit us to say a few words concerning the ancient author of this work, and its able translator, Dr. Robert Simson, of the *University of Glasgow*, but our readers may judge of the accuracy of an edition hitherto read in all our schools and universities, which, though it has passed through the press many times, declares on its title-page that Dr. Robert Simson belonged to the *University of Oxford.*

EUCLID'S

ELEMENTS OF GEOMETRY.

BOOK I.

DEFINITIONS.

I.

A POINT is that which hath no parts, or which hath no magnitude.

"A point is" more clearly defined to be "the beginning of magnitude;" as, for instance, the beginning of a line.

II.

A line is length without breadth.

A line is extension in any one direction, uniform or variable; as, the unbroken contour or outline of any given surface.

III.

The extremities of a line are points.

By the extremities of a line, are here meant, the beginning and the end of the line.

IV.

A straight line is that which lies evenly between its extreme points.

"A straight line is" more clearly defined to be "that in which, if any two points be taken, the part intercepted between them is the shortest that can be drawn." This shows that every straight line in the Elements is considered to be of indefinite length, unless otherwise expressed.

V.

A superficies is that which hath only length and breadth.

A superficies or surface, is extension in any two directions, uniform or variable; as, the continuous boundary of any given solid. Def. I. Book XI.

VI.

The extremities of a superficies are lines.

By the extremities of a superficies or surface, are here meant, the boundaries or edges of the surface.

VII.

A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.

A plane superficies, or simply, a plane, is a surface in which a straight line can any where be drawn. This shows that every plane in the Elements is considered to be of indefinite extent, unless otherwise expressed.

VIII.

[A plane angle is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.]

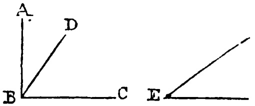
This definition is put in brackets, as useless, and unnecessary to be remembered.

IX.

A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

When two straight lines meet at a point, so that if produced they would intersect (cross) each other, the indefinite space between them is called an angle.

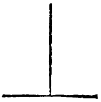
N.B. "When several angles are at one point B, any one of them is expressed by three letters, of which the letter that is at the vertex of the angle, that is, at the point in which the straight lines that contain the angle meet one another, is put between the other two letters, and one of these two is somewhere upon one of those straight lines, and the other upon the other straight line: thus the angle which is contained by the straight lines, AB, CB, is named the angle ABC, or CBA; that which is contained by AB, DB, is named the angle ABD, or DBA; and that which is contained by DB, CB, is called the angle DBC, or CBD; but, if there be only one angle at a point, it may be expressed by a letter placed at that point; as the angle at E."



This explanation is put in inverted commas, as being Dr. Simson's addition; it is very necessary to be remembered.

X.

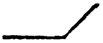
When a straight line standing on another straight line makes the adjacent angles equal to one another, each of these angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.



By borrowing the terms vertical and horizontal from the language of Physics, we may define a right angle to be that which is formed by the meeting of a vertical and horizontal line.

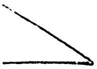
XI.

An obtuse angle is that which is greater than a right angle.



XII.

An acute angle is that which is less than a right angle.



The term oblique angles is applied both to obtuse angles and to acute angles.

XIII.

[A term or boundary is the extremity of any thing.]

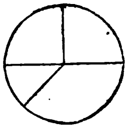
This definition is unnecessary, being merely verbal.

XIV.

A figure is that which is enclosed by one or more boundaries.

This definition is applicable to solid figures, as well as to plane figures.

XV.



A circle is a plane figure contained [or, bounded] by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference, are equal to one another.

The circumference of a circle is its boundary. The space contained within the boundary, is called the circle.

XVI.

And this point is called the centre of the circle.

The straight line drawn from the centre to the circumference, is called the radius.

XVII.

A diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

XVIII.

A semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

XIX.

[A segment of a circle is the figure contained by a straight line, and the part of the circumference it cuts off.]

This definition is repeated in Book III. Definition VI.

XX.

Rectilineal [or, rectilinear, that is, *formed of straight lines*,] figures are those which are contained [or, bounded] by straight lines.

XXI.

Trilateral [that is, *three-sided*] figures, or triangles, [that is, *three angled*, are those which are contained] by three straight lines.

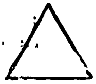
XXII.

Quadrilateral, [that is, *four-sided;* or, quadrangles, that is, *four angled*, are those which are contained] by four straight lines.

XXIII.

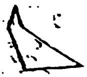
Mutilateral [that is, *many-sided*] figures, or polygons, [that is, *many angled*, are those which are contained] by more than four straight lines.

XXIV.



Of three-sided figures, an equilateral [that is, *equal-sided*] triangle is that which has three equal sides.

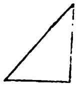
XXV.



An isosceles [that is, *equal-legged*] triangle is that which has only two sides equal.

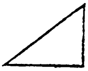
XXVI.

A scalene [that is, *unequal*] triangle is that which has three unequal sides.



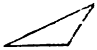
XXVII.

A right-angled triangle is that which has a right angle.



XXVIII.

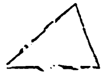
An obtuse-angled triangle is that which has an obtuse angle.



XXIX.

An acute-angled triangle is that which has three acute-angles.

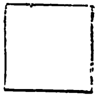
The acute angled triangle must have *three* acute angles, because the two preceding species of triangles have each *two* acute angles, as will be shown in the sequel.



XXX.

Of four-sided figures, a square is that which has all its sides equal, and all its angles right angles.

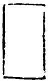
This definition is redundant. If the general definition annexed to the 34th Prop. of this book be considered, the square is only a species of parallelogram, viz. that which has one angle a right angle and the sides which contain it equal to one another.



XXXI.

An oblong is that which has all its angles right angles, but has not all its sides equal.

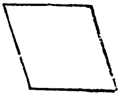
This definition is also redundant; for an oblong [or rectangle, that is, a right-angled parallelogram] is that which has one angle a right angle, and the sides which contain it equal.



XXXII.

A rhombus is that which has all its sides equal, but its angles are not right angles.

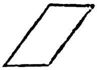
This is also redundant; for a rhombus is that which has one angle oblique, and the sides which contain it unequal.



XXXIII.

A rhomboid is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.

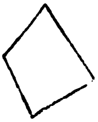
This definition may be replaced by that of a parallelogram above mentioned.



XXXIV.

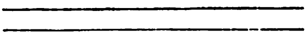
All other four-sided figures besides these, are called Trapeziums.

Quadrilateral figures whose opposite sides are not parallel, are called trapeziums; but if one opposite pair be parallel and the *other pair not*, the figure is called a trapezoid.



XXXV.

Parallel straight lines are such as are in the same plane, and which being produced ever so far both ways do not meet.



The meaning of this definition is, that the space between the lines is always of the same breadth.

POSTULATES.

I.

Let it be granted, that a straight line may be drawn from any one point to any other point.

When a straight line is drawn from one point to another point, the points are said to be *joined*. The points are understood to be in the same plane.

II.

That a terminated straight line may be produced to any length in a straight line.

By *terminated* here, is meant of a *definite* length; and by *produced* is meant lengthened or extended *indefinitely*, in the same plane.

III.

And that a circle may be described from any centre, at any distance from that centre.

By describing a circle *at any distance*, is meant drawing a circle in a plane with any given radius.

Various other postulates (that is, *demands* of common sense,) are tacitly assumed by Euclid; as, that one figure or angle may be applied to another, for the purpose of comparison. See Prop. IV. of this book.

AXIOMS.

I.

Things which are equal to the same thing, are equal to one another.

II.

If equals be added to equals, the wholes are equal.

III.

If equals be taken from equals, the remainders are equal.

IV.

If equals be added to unequals, the wholes are unequal.

V.

If equals be taken from unequals, the remainders are unequal.

VI.

Things which are double of the same, are equal to one another.

VII.

Things which are halves of the same, are equal to one another.

VIII.

Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

IX.

The whole is greater than its part.

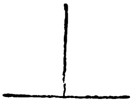
Dr. Thomson, in his edition of Euclid, has added to this axiom, another, viz., that "the whole is equal to all its parts taken together."

X.

Two straight lines cannot enclose a space.

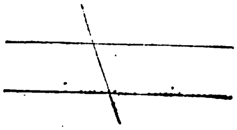
XI.

All right angles are equal to one another.



XII.

If a straight line meets two straight lines, so as to make the two interior angles on the same side of it, taken together, less than two right angles, these straight lines, being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles.



It is admitted by Dr. Simson that this axiom is not *self-evident*, which all axioms ought to be. Accordingly, he demonstrates the truth of it as a proposition in his notes, by help of *five* different propositions! Though not considered free from objection, the substitute for this axiom, given in Playfair's edition of Euclid, is to be preferred: viz., "If two straight lines intersect each other, they cannot be both parallel to the same straight line."

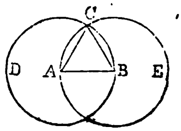
The number of axioms is in this book limited to twelve; but Euclid has tacitly assumed the truth of various other axioms, which will be noticed in the sequel.

PROP. I. PROBLEM.

To describe an equilateral triangle upon a given finite straight line.

Let AB be the given straight line. It is required to describe an equilateral triangle upon AB.

From the centre A, at the distance AB, describe (*Post.* 3) the circle BCD. From the centre B, at the distance BA, describe the circle ACE. And from the point C, in which the circles cut one another, draw the straight lines (*Post.* 1) CA, CB, to the points A, B. Then ABC is an equilateral triangle.



Because the point A is the centre of the circle BCD, AC is equal (*Def.* 15) to AB. And because the point B is the centre of the circle ACE, BC is equal to BA. But it has been proved that CA is equal to AB. Therefore the two straight lines CA, CB, are each of them equal to AB. But things which are equal to the same thing are equal (*Ax.* 1) to one another. Therefore CA is equal to CB. Wherefore the three sides CA, AB, BC, are equal to one another. The triangle ABC is, therefore, equilateral. And it is described upon the given straight line AB. Q. E. F.

From the construction of this problem, it is plain that, upon any given straight line, two equilateral triangles may be described, viz. one on each side.

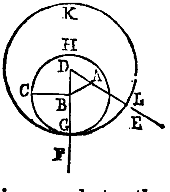
Exercise. To describe an isosceles triangle upon a given finite straight line, that shall have each of its sides double the base.

PROP. II. PROBLEM.

From a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line. It is required to draw from the point A a straight line equal to BC.

From the point A to B draw (*Post.* 1) the straight line AB. Upon AB describe (I. 1) the equilateral triangle DAB. And produce (*Post.* 2) the straight lines DA, DB, to the points E and F. From the centre B, at the distance BC, describe (*Post.* 3) the circle CGH. And from the centre D, at the distance DG, describe the circle GKL. Then, the straight line AL is equal to BC.



Because the point B is the centre of the circle CGH, BC is equal (*Def.* 15) to BG. And because the point D is the centre of the circle GKL, DL is equal to DG. But (*Const.*) DA, DB, parts of these equals, are equal. Therefore the remainder AL is equal to the remainder (*Ax.* 3) BG. But it has been shown that BC is equal to BG. Wherefore AL and BC are each of them equal to BG. And things that are equal to the same thing are equal (*Ax.* 1) to one another. Therefore the straight line AL is equal to the straight line BC. Wherefore from the given point A, a straight line AL has been drawn equal to the given straight line BC. Q. E. F.

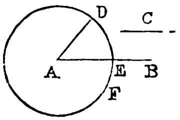
The construction of this problem might be improved thus:—Join AB. Upon AB describe the equilateral triangle ABD. From the centre B, at the distance

B C, describe the circle CGH. Produce DB to meet the circumference in G. From the centre D, at the distance DG, describe the circle GKL. Produce DA to meet the circumference in L. Then AL is equal to BC. The demonstration of this construction will be the same as that above.

Exercise. Draw the figures, and show the application of the construction and demonstration to different positions of the point and the straight line; such as, when the given point is situated above the straight line or below the straight line; also, when in the straight line itself, at the extremities, or at any point between them.

PROP. III. PROBLEM.

From the greater of two given straight lines to cut off a part equal to the less.



Let AB and C be the two given straight lines, of which AB is the greater. It is required to cut off from AB, the greater, a part equal to C, the less.

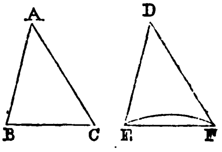
From the point A draw (I. 2) the straight line AD equal to C. And from the centre A, at the distance AD, describe (*Post.* 3) the circle DEF. Then the part AE shall be equal to C.

Because A is the centre of the circle DEF, AE is equal (*Def.* 15) to AD. But the straight line C is likewise equal (*Const.*) to AD. Therefore AE and C are each of them equal to AD. Wherefore the straight line AE is equal (*Ax.* 1) to C. Therefore from AB the greater of two given straight lines, a part AE has been cut off equal to C the less. Q. E. F.

Exercise. To produce the smaller of two given straight lines, so that with the part produced, it shall be equal to the greater.

PROP. IV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by these sides equal to one another, their bases, or third sides, *are equal; the two triangles are equal; and their other angles are equal, each to each; viz., those to which the equal sides are opposite.*



Let ABC, DEF be two triangles, which have the two sides AB, AC equal to the two sides DE, DF, each to each; viz., AB to DE, and AC to DF; and the angle BAC equal to the angle EDF. Then the base BC is equal to the base EF; the triangle ABC is equal to the triangle DEF; and the remaining angles of the one are equal to the remaining angles of the other, each to each; viz., those to which the equal sides are opposite; that is, the angle ABC is equal to the angle DEF, and the angle ACB to the angle DFE.

For, if the triangle ABC be applied to the triangle DEF, so that the point A may be on the point D, and the straight line AB upon the straight line DE. The point B shall coincide (that is, fall upon, so as to agree) with the point E, because AB is equal (*Hyp.*) to DE. And AB coinciding with DE, AC shall coincide with DF, because the angle

BAC is equal (*Hyp.*) to the angle EDF. Also, the point C shall coincide with the point F, because AC (*Hyp.*) is equal to DF. But the point B was proved to coincide with the point E. Therefore the base BC shall coincide with the base EF. For, the point B coinciding with the point E, and the point C with the point F, if the base BC does not coincide with the base EF, the two straight lines BC, EF, would enclose a space, which (*Ax.* 10) is impossible. Wherefore the base BC coincides with the base EF, and is, therefore, equal (*Ax.* 8) to it. Wherefore, also, the whole triangle ABC coincides with the whole triangle DEF, and is, therefore, equal to it. And the remaining angles of the one coincide with the remaining angles of the other, and are therefore equal to them, each to each; viz., the angle ABC to the angle DEF, and the angle ACB to the angle DFE. Therefore, if two triangles have two sides, &c. Q. E. D.

The demonstration of this proposition might have been conducted by beginning the application of the one triangle to the other at the points B and E, and then going round the figure as above.

This proposition holds equally true, when the angles contained by the two sides, of the one triangle is the *same* as that contained by the two sides of the other; as, in the triangles FAC, GAB, see fig. to next proposition; or, when the triangles have a *common* base, as in the triangles FBC, GCB, see the same fig.; or when they have a common side.

Corollary. If the sides AB, DE, or the sides AC, DF, were produced, it would be shown in the same manner, that the angles formed upon the other sides of the bases of the triangles would be equal, each to each.

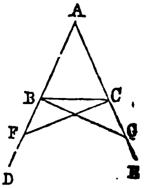
Exercise. If two squares have one side of the one equal to one side of the other, the squares are equal in all respects.

PROP. V. THEOREM.

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles upon the other side of the base shall be equal.

Let ABC be an isosceles triangle, of which the side AB is equal to the side AC. And let the equal sides AB, AC be produced to D and E. Then, the angle ABC is equal to the angle ACB, and the angle CBD to the angle BCE.

In BD take any point F, and from AE the greater, cut off AG equal (I. 3) to AF the less. Join FC, GB.



Because in the two triangles AFC, AGB, AF is equal to (*Const.*) AG, and AB to (*Hyp.*) AC, the two sides FA, AC are equal to the two sides GA, AB, each to each; and they contain the angle FAG common to the two triangles AFC, AGB. Therefore the base FC is equal (I. 4) to the base GB, and the triangle AFC to the triangle AGB. Also, the remaining angles of the one are equal (I. 4) to the remaining angles of the other, each to each; viz., those to which the equal sides are opposite; that is, the angle ACF to the angle ABG, and the angle AFC to the angle AGB. Again, because the whole AF is equal to the whole AG, of which the parts AB, AC, are equal. Therefore the remainder BF is equal (*Ax.* 3) to the remainder CG. But, FC was proved to equal to GB. Therefore, in the two triangles BFC, GCB, the two sides BF, FC are equal to the two sides CG, GB, each to each. And the angle BFC was proved to be equal to the angle CGB. Wherefore the two triangles BFC, CGB, are equal (I. 4), and their remaining angles are equal, each to each, viz., those to which the equal sides are opposite. Therefore the angle FBC is equal to the angle GCB, and the angle BCF to the angle CBG. But it has been demonstrated, that the whole angle ABG is equal to the whole angle ACF, and the part CBG of the one, equal to the part BCF of the other. Therefore the remaining angle ABC is equal (*Ax.* 3) to the remaining angle ACB, and these are the angles at the base of the triangle ABC. It has also been proved that the angle FBC is equal to the angle GCB, and these are the angles upon the other side of the base. Therefore the angles at the base, &c. Q. E. D.

Corollary. Hence every equilateral (equal-sided) triangle is also equiangular (equal-angled).

This demonstration might be shortened by the application of the *corollary* to the preceding proposition. To do so, will be a useful exercise to the student.

The enunciation of this proposition is more clearly expressed thus: "*If two sides of a triangle be equal to one another, the angles which are opposite to the equal sides, are also equal to one another;*" and if the equal sides be produced, the angles upon the other side of the base shall likewise be equal.

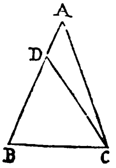
PROP. VI. THEOREM.

If two angles of a triangle be equal to one another, the sides which subtend, or are opposite to *the equal angles, are equal to one another.*

Let ABC be a triangle having the angle ABC equal to the angle ACB. Then the side AB is equal to the side AC.

For, if AB be not equal to AC, one of them is greater than the other Let AB be the greater; and from BA cut (I. 3) off BD equal to AC, the less. And join DC.

Because in the two triangles DBC, ACB, the side DB is equal to the side AC, and BC is common to both, the two sides, DB, BC are equal to the two sides AC, CB, each to each. And the angle DBC is equal to the angle (*Hyp.*) ACB. Therefore the base DC is equal to the base AB. And the triangle DBC is equal to the triangle (I. 4) ACB, the less to the greater, which is absurd. Therefore AB is not unequal to AC; that is, AB is equal to AC. Wherefore, if two angles, &c. Q. E. D.



Corollary. Hence every equiangular triangle is also equilateral.

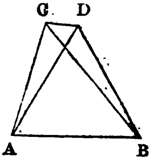
This proposition is called the converse of Prop. V., because the *hypothesis* of that proposition is the *predicate* of this one.

Exercise. In the figure of Prop. V., let a straight line be drawn from the point A to the point of intersection of the two straight lines BG, FC. It is required to prove that this straight line will bisect the angle FAG. This suggests a readier mode of bisecting an angle than that contained in Prop. IX. of this book.

PROP. VII. THEOREM.

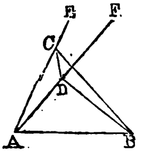
Upon the same base, and on the same side of it, there cannot be two triangles having their sides terminated in one extremity of the base, equal to one another, and likewise those terminated in the other extremity.

If it be possible, upon the same base AB, and upon the same side of it, let there be two triangles ACB, ADB, having their sides CA, DA terminated in the extremity A of the base, equal to one another, and likewise their sides, CB, DB, terminated in the extremity B.



Join CD. First, let the vertex of each triangle be without the other triangle. Because AC is equal (*Hyp.*) to AD in the triangle ACD, the angle ACD is equal (I. 5) to the angle ADC. But the angle ACD is greater (*Ax.* 9) than the angle BCD. Therefore the angle ADC is also greater than the angle BCD. Much more then is the angle BDC greater than the angle BCD. Again, because CB is equal (*Hyp.*) to DB, the angle BDC is equal (I. 5) to the angle BCD. But it has been demonstrated that the angle BDC is greater than the angle BCD. Therefore, the angle BDC is both equal to, and greater than the angle BCD; which is impossible.

Secondly, let the vertex of one of the triangles be within the other triangle. Produce AC and AD to E and F. Because AC is equal (*Hyp.*) to AD in the triangle ACD, the angles ECD, FDC upon the other side of the base CD are equal (I. 5) to one another. But the angle ECD is greater (*Ax.* 9) than the angle BCD. Therefore the angle FDC is likewise greater than BCD. Much more then is the angle BDC greater than the angle BCD. Again, because CB is equal (*Hyp.*) to DB, the angle BDC is equal (I. 5) to the angle BCD. But it has been proved that the angle BDC is greater than the angle BCD. Therefore, the angle BDC is both equal to, and greater than the angle BCD, which is impossible.



The third case, in which the vertex of one triangle is upon a side of the other needs no demonstration.

Therefore, upon the same base, and on the same side of it, &c. Q. E. D.

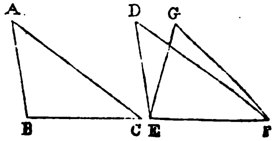
The third case needs no demonstration, because when the vertex of the one triangle is upon a side of the other, that is, on a point between A and C, in the triangle ACB, the sides which terminate in one extremity of the base are unequal, which is contrary to the hypothesis.

Exercise. If two triangles on the same base, and on opposite sides of it, have their sides terminated in one extremity of the base equal, and likewise those terminated in the other extremity; the angle contained by the two sides of the one shall be equal to the angle contained by the two sides of the other.

PROP. VIII. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; the angle which is contained by the two sides of the one is equal to the angle contained by the two sides equal to them, of the other.

Let ABC, DEF be two triangles, having the two sides AB, AC equal to the two sides DE, DF, each to each; viz., AB to DE, and AC to DF; and also the base BC equal to the base EF. The angle BAC is equal to the angle EDF.



For, if the triangle ABC be applied to the triangle DEF, so that the point B shall be on E, and the base BC upon the base EF; then, the point C shall coincide with the point F, because BC is equal (*Hyp.*) to EF. And BC coinciding with EF, BA and AC shall coincide with ED and DF. For, if the base BC coincides with the base EF, and the sides BA, AC do not coincide with the sides ED, DF, but have a different situation as EG, GF. Then, upon the same base EF, and upon the same side of it, there can be two triangles having their sides terminated in one extremity of the base equal to one another, and likewise those terminated in the other extremity. But this is (I. 7) impossible. Wherefore, if the base BC coincides with the base EF, the sides BA, AC cannot but coincide with the sides ED, DF. Therefore, the angle BAC coincides with the angle EDF, and is equal (*Ax.* 8) to it. Therefore if two triangles, &c. Q. E. D.

Corollary. It is a plain inference from the demonstration of this proposition, that if the three sides of one triangle be equal to the three sides of another triangle, each to each; the three angles of the one shall be equal to the three angles of the other, each to each,—viz., those to which the equal sides are opposite.

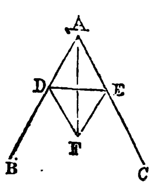
Pappus demonstrates this proposition without the aid of Prop. VII., by applying the base of the one triangle to the base of the other, and inverting the former, so that they shall correspond exactly to the terms of the proposition contained in the exercise appended to the preceding proposition.

PROP. IX. PROBLEM.

To bisect a given rectilineal angle; that is, to divide it into two equal angles.

Let BAC be the given rectilineal angle; it is required to bisect it.

Take any point D in AB, and from AC cut off AE (I. 3) equal to AD. Join DE. Upon DE, opposite to the triangle DAE, describe (I. 1) an equilateral triangle DEF. Join AF. The straight line AF bisects the angle BAC.



Because AD is equal (*Const.*) to AE, and AF is common to the two triangles DAF, EAF, the two sides DA, AF, are equal to the two sides EA, AF, each to each; and the base DF is equal (*Const.*) to the base EF. Therefore the angle DAF is equal (I. 8) to the angle EAF. Wherefore the given rectilineal angle BAC is bisected by the straight line AF. Q. E. F.

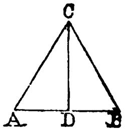
By means of this proposition, an angle may be divided into any number of equal parts denoted by the successive powers of the number 2; that is, into 2, 4, 8, 16, 32, 64, &c., equal parts; but to divide any angle into *three* equal parts, that is, to trisect an angle, is beyond the power of elementary geometry.

PROP. X. PROBLEM.

To bisect a given finite straight line: that is, to divide it into two equal parts.

Let AB be the given straight line. It is required to divide it into two equal parts.

Describe upon AB (I. 1) an equilateral triangle ABC, and bisect (I. 9) the angle ACB by the straight line CD, meeting AB in the point D. Then, AB is divided into two equal parts at the point D.



Because AC is equal (*Const.*) to CB, and CD common to the two triangles ACD, BCD, the two sides AC, CD, are equal to the two sides BC, CD, each to each; and the angle ACD is equal (*Const.*) to the angle BCD. Therefore the base AD is equal to the base (I. 4) DB, and the straight line AB is divided into two equal parts at the point D. Q. E. F.

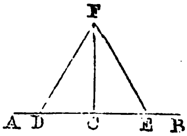
By this problem a straight line may be divided into any number of equal parts denoted by the series 2, 4, 8, 16, &c.

PROP. XI. PROBLEM.

To draw a straight line at right angles to a given straight line, from a given point in the same.

Let AB be a given straight line, and C a given point in it. It is required to draw a straight line from the point C at right angles to AB.

Take any point D in AC, and make (I. 3) CE equal to CD. Upon DE describe (I. 1) the equilateral triangle DFE, and join FC. The straight line FC drawn from the given point C, is at right angles to the given straight line AB.

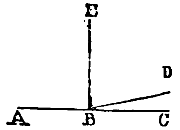


Because DC is equal (*Const.*) to CE, and FC common to the two triangles DCF, ECF, the two sides DC, CF, are equal to the two sides EC, CF, each to each; and the base DF is equal (*Const.*) to the base EF. Therefore the angle DCF is equal (I. 8) to the angle ECF: and they are adjacent angles. But when the adjacent angles which one straight line makes with another straight line, are equal to one another, each of them is called a right (*Def.* 10) angle. Therefore each of the angles DCF, ECF, is a right angle. Wherefore, from the given point C, in the given straight line AB, a straight line FC has been drawn at right angles to AB. Q. E. F.

COR.—Two straight lines cannot have a common segment; that is, they cannot coincide in part without coinciding altogether.

If it be possible, let the two straight lines ABC, ABD, have the segment AB common to both of them.

From *the point* B draw (I. 11) BE at right angles to AB.



Because A B C is a straight line, the angle CBE is equal (*Def.* 10) to the angle EBA. Because ABD is a straight line, the angle DBE is equal to the angle EBA. Therefore (*Ax.* 1) the angle DBE is equal to the angle CBE, the less to the greater, which is impossible. Therefore two straight lines cannot have a common segment. Q. E. D.

In Playfair's edition of the Elements, this corollary is drawn from the defective definition of a straight line, which he has substituted for Euclid's.

Exercises.—1. From the extremity of a given straight line, to draw a straight line which shall make a right angle with it.

2. In a straight line of unlimited length, given in position, to find a point equally distant from two given points.

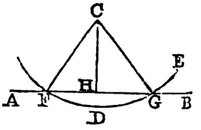
3. If the three sides of a triangle be bisected, and straight lines be drawn through the points of bisection at right angles to the sides, they shall, if produced, meet in the same point.

4. To describe a square upon a given straight line.

PROP. XII. PROBLEM.

To draw a straight line perpendicular to a given straight line of unlimited length, from a given point without it.

Let AB be the given straight line, which may be produced to any length both ways; and let C be a point without it.



It is required to draw a straight line perpendicular to A B from the point C.

Take any point D upon the other side of AB, and from the centre C, at the distance CD, describe (*Post.* 3) the circle EGF meeting AB in F and G. Bisect (I. 10) FG in H, and join CH. The straight line CH, drawn from the given point C, is perpendicular to the given straight line AB.

Join CF, CG. Because FH is equal (*Const.*) to HG, and HC common to the two triangles FHC, GHC, the two sides FH, HC, are equal to the two sides GH, HC, each to each; and the base CF is equal (*Def.* 15) to the base CG. Therefore the angle CHF is equal (I. 8) to the angle CHG; and they are adjacent angles. But when a straight line standing on another straight line makes the adjacent angles equal to one another, each of them is a right angle, and the straight line which stands upon the other is called a perpendicular (*Def.* 10) to it. Therefore from the given point C a perpendicular CH has been drawn to the given straight line AB. Q. E. D.

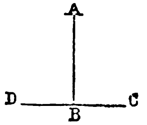
Exercise. From a point nearly in the perpendicular to a straight line at one extremity, but within the right angle, to draw a straight line which shall make a right angle with it.

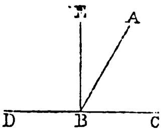
PROP. XIII. THEOREM.

The angles which one straight line makes with another upon one side of it, are either two right angles, or are together equal to two right angles.

Let the straight line AB make with CD, upon one side of it, the two angles CBA, ABD. These angles are either two right angles, or are together equal to two right angles.

For if the angle C B A be equal to the angle A B D, each of them is a right angle (*Def.* 10). But if the angles C B A, A B D, be unequal, from the point B draw B E at right angles (I. 11) to C D. Therefore the angles C B E, E B D (*Def.* 10) are two right angles. But the angle C B E is equal to the two angles C B A, A B E, together. To each of these equals add the angle E B D. Therefore the two angles C B E, E B D, are equal (*Ax.* 2) to the three angles C B A, A B E, E B D. Again, the angle D B A is equal to the two angles D B E, E B A. To each of these equals add the angle A B C. Therefore the two angles D B A, A B C, are equal (*Ax.* 2) to the three angles D B E, E B A, A B C. But the two angles C B E, E B D, have been proved to be equal to the same three angles. And things that are equal to the same thing are equal (*Ax.* 1) to one another. Therefore the two angles C B E, E B D, are equal to the two angles D B A, A B C. But the two angles C B E, E B D, are two right angles. Therefore the two angles D B A, A B C, are together equal (*Ax.* 1) to two right angles. Wherefore, the angles which one straight line, &c. Q. E. D.





Corollary 1.—All the angles made by any number of straight lines meeting at a point, on one side of a straight line, are together equal to two right angles.

Corollary 2.—All the angles made by any number of straight lines meeting in a point, are together equal to four right angles.

Definition 1.—When two angles are together equal to two right angles, the one is called the *supplement* of the other.

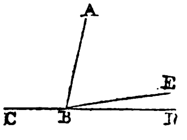
Definition 2.—When two angles are together equal to a right angle, the one is called the *complement* of the other.

Exercise.—If an angle and its supplement be bisected, the bisecting lines are perpendicular to each other.

PROP. XIV. THEOREM.

If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the two adjacent angles together equal to two right angles, these two straight lines are in one and the same straight line.

At the point B, in the straight line AB, let the two straight lines BC, BD upon the opposite sides of AB, make the adjacent angles ABC, ABD together equal to two right angles. Then, B D is in the same straight line with C B.



For, if BD be not in the same straight line with CB, let B E be in the same straight line with it. Because the straight line A B makes with the straight line CBE, upon one side of it, the two angles ABC, ABE, these two angles are together equal (I. 13) to two right angles. But the two angles ABC, ABD are likewise together equal (*Hyp.*) to two right angles. Therefore the two angles CBA, ABE are equal (*Ax.* 1) to the two angles CBA, ABD. From each of these equals, take away the common angle A B C. Therefore the remaining angle AB E is equal (Ax. 3) to the remaining angle AB D, the less to the greater, which is impossible.

Wherefore BE is not in the same straight line with BC. In like manner, it may be shown that no other straight line but B D, can be in the same straight line with B C. Therefore BD is in the same straight line with CB. Wherefore, if at a point, &c. Q. E. D.

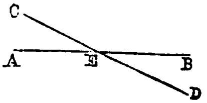
Corollary.—If at a point in a straight line, two other straight lines upon the same side of it, make each a right angle with it, these two straight lines shall coincide with each other.

PROP. XV. THEOREM.

If two straight lines cut one another, the vertical, or opposite angles are equal.

Let the two straight lines A B, CD, cut one another in the point E. The angle AEC is equal to the angle DEB, and the angle C E B to the angle A E D.

Because the straight line A E makes with CD the two angles C E A, AED, these angles are together equal (I. 13) to two right angles. Again, because the straight line D E makes with A B the two angles AED, DEB, these angles are together equal (I. 13) to two right angles. But the two angles CEA, AED, have been proved to be equal to two right angles. Therefore the two angles CEA, AED, are equal (*Ax.* 1) to the two angles AED, DEB. From these equals, take away the common angle AED. Therefore the remaining angle CEA is equal (*Ax.* 3) to the remaining angle D E B. In the same manner, it can be demonstrated, that the angle CEB is equal to the angle AED. Therefore, if two straight lines, &c. Q. E. D.



The corollaries added to this proposition by Euclid, are included in those now added to Prop. XIII.

Exercise 1.—If at a point in a straight line two other straight lines meet upon the opposite sides of it, and make the vertical or opposite angles equal, these two straight lines are in one and the same straight line.

Exercise 2.—If two straight lines cut one another, and the vertical angles be bisected, the bisecting lines are in one and the same straight line.

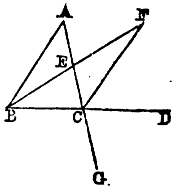
PROP. XVI. THEOREM.

If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

Let A B C be a triangle, and let its side B C be produced to D. The exterior angle ACD is greater than either of the interior opposite angles CBA, BAC.

Bisect (I. 10) A C in E, join BE and produce it to F. Make E F equal (I. 3) to B E. Join F C.

Because AE is equal (*Const.*) to EC, and B E (*Const.*) to E F. Therefore, in the triangles A E B, C E F, the the two sides A E, E B, are equal to the two sides CE, EF, each to each. But the angle AEB is equal (I. 15) to the angle CEF, because they are vertical angles. Therefore the base AB is equal (I. 4) to the base CF, the triangle AEB to the triangle CEF, and the remaining angles of the one to the remaining angles of the other, each to each:—viz., those to which



the equal sides are opposite. Wherefore the angle B A E is equal to the angle E C F. But the angle E C D is greater (*Ax.* 9) than the angle E C F. Therefore the angle A C D is greater than the angle B A E. In the same manner, if the side B C be bisected, and A C be produced to G, it may be demonstrated that the angle B C G, is greater than the angle A B C. But the angle A C D is equal (I. 15) to the angle B C G. Therefore the angle A C D is greater than the angle A B C. Therefore, if one side, &c. Q. E. D.

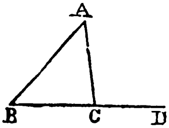
The student should for the sake of practice, write out the demonstration of the second part here alluded to; otherwise, the truth of the proposition will not be so completely fixed in his mind. A new axiom is taken for granted in the demonstration of this, and some subsequent propositions; viz., If two things be equal to one another, and the one be greater than a third, so is the other.

Exercise.—From a point without a straight line, only one perpendicular can be drawn to it.

PROP. XVII. THEOREM.

Any two angles of a triangle are together less than two right angles.

Let A B C be any triangle; any two of its angles are together less than two right angles.



Produce B C to D. Because A C D is the exterior angle of the triangle A B C, the angle A C D is greater (I. 16) than the interior and opposite angle A B C. To each of these unequals, add the angle A C B. Therefore the two angles A C D, A C B, are greater (*Ax.* 4) than the two angles A B C, A C B. But the two angles A C D, A C B are together equal (I. 13) to two right angles. Therefore the two angles A B C, B C A are together less than two right angles. In like manner, it may be demonstrated, that the two angles B A C, A C B, as also the two angles C A B, A B C, are together less than two right angles. Therefore any two angles, &c. Q. E. D.

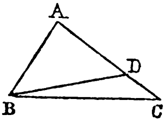
Exercise 1.—The three interior angles of any triangle are less than three right angles.

Exercise 2.—The two exterior angles of any triangle are greater than two right angles; and the three exterior angles are greater than three right angles.

PROP. XVIII. THEOREM.

The greater side of every triangle is opposite to the greater angle.

Let A B C be a triangle, of which the side A C is greater than the side A B. The angle A B C is greater than the angle B C A.



From A C cut off (I. 3) A D equal to A B. Join B D.

Because A D B is the exterior angle of the triangle B D C, it is greater (I. 16) than the interior and opposite angle D C B. But the angle A D B is equal (I. 5) to the angle A B D, because the side A B is equal (*Const.*) to the side A D. Therefore the angle A B D is likewise greater than the angle A C B. Much more, therefore, is the angle A B C greater than the angle A C B. Therefore the greater side, &c. Q. E. D.

Corollary.—One angle of a triangle is greater than, equal to, or less than another, according as the side opposite to the former is greater than, equal to, or less than the side opposite to the latter.

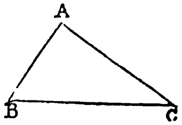
Corollary 2.—All the angles of a scalene triangle are unequal.

PROP. XIX. THEOREM.

The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

Let A B C be a triangle, of which the angle A B C is greater than the angle B C A. The side A C is greater than the side A B.

For, if the side A C be not greater than the side A B, A C must either be equal to A B, or less than it. The side A C is not equal to the side A B. Because then the angle A B C would be equal (I. 5) to the angle A C B. But it is (*Hyp.*) not. Therefore A C is not equal to A B. The side A C is not less than the side A B. Because then the angle A B C would be less (I. 18) than the angle A C B. But it is (*Hyp.*) not. Therefore the side A C is not less than A B. And it has been shown that A C is not equal to A B. Therefore the side A C is greater than the side AB. Wherefore the greater angle, &c. Q. E. D.



Corollary.—One side of a triangle is greater than, equal to, or less than another, according as the angle opposite to the former, is greater than, equal to, or less than the angle opposite to the latter.

Exercise.—If from a point without a given straight line, any number of straight lines be drawn to meet it; of all these straight lines, that which is perpendicular to the given straight line is the least; and of others that which is nearer to the perpendicular is always less than the more remote; also, from the same point only two equal straight lines can be drawn to the given straight line, one upon each side of the shortest line.

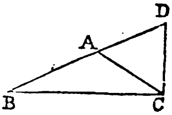
PROP. XX. THEOREM.

Any two sides of a triangle are together greater than the third side.

Let A B C be a triangle: any two of its sides are together greater than the third side; viz., the sides B A, A C, are greater than the side B C; the sides A B, B C greater than A C; and the sides B C, C A greater than A B.

Produce B A to the point D, and make (I. 3) A D equal to A C. Join D C.

Because D A is equal to A C, the angle A D C is equal (I. 5) to the angle A C D. But the angle B C D is greater (*Ax.* 9) than the angle A C D: therefore the angle B C D is greater than the angle A D C. But because the angle B C D of the triangle D C B is greater than its angle B D C, and the greater (I. 19) angle is subtended by the greater side. Therefore the side D B is greater than the side B C. Again A D is equal (*Const.*) to A C. To each of these equals, add B A. Therefore the whole B D is equal (*Ax.* 2) to the two B A and A C. But B D was proved to be greater than B C. Therefore the sides B A, A C are greater than B C. In the same manner it may be demonstrated, that the two sides A B,



B C are greater than C A, and the two sides B C, C A greater than AB. Therefore any two sides, &c. Q. E. D.

This proposition is a corollary to the definition of a straight line, given in the annotation to Def. 4.

Exercise 1.—Any side of a triangle is greater than the difference between the other two sides.

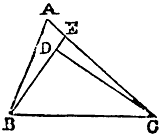
Exercise 2.—The three sides of a triangle taken together are greater than the double of any one side, but less than the double of any two sides.

Exercise 3.—From two given points on the same side of a straight line given in position, to draw two straight lines which shall meet at a point in it, and which taken together shall be less than the sum of two straight lines drawn from the same points to any other point in the given straight line.

PROP. XXI. THEOREM.

If from the ends of one side of a triangle, there be drawn two straight lines to a point within the triangle, these together shall be less than the other two sides of the triangle, but shall contain a greater angle.

Let A B C be a triangle, and from the points B, C, the ends of the side B C, let the two straight lines B D, C D be drawn to the point D within the triangle. Then B D and D C together shall be less than the other two sides B A, A C of the triangle A B C, but shall contain an angle B D C greater than the angle B A C.



Produce B D to E. Then, the two sides B A, A E, of the triangle A B E are greater than the third side B E (I. 20). To each of these unequals add E C. Therefore the two sides B A, A C are greater (*Ax.* 4) than B E, E C. Again, the two sides C E, E D of the triangle C E D are greater (I. 20) than the third side C D. To each of these unequals, add D B. Therefore the two sides C E, E B, are greater (*Ax.* 4) than C D, D B. But it has been shown that B A, A C are greater than B E, E C. Much more then are B A, A C, greater than B D, D C.

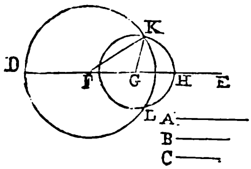
Again, the exterior angle B D C of the triangle C D E is greater than its interior and opposite angle C E D (I. 16). And the exterior angle C E B of the triangle A B E is greater than its interior and opposite angle B A C (I. 16). But the angle B D C is greater than the angle C E B. Much more then is the angle B D C greater than the angle B A C. Therefore, if from the ends of, &c. Q. E. D.

Exercise. If from any point within a triangle, straight lines be drawn to the vertices of the three angles, these three straight lines taken together shall be less than the sum of the three sides, but greater than half that sum.

PROP. XXII. PROBLEM.

To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of them must be greater than the third (I. 20).

Let A, B, C, be the given straight lines, of which any two whatever are greater than the third; viz., A and B greater than C; A and C greathan B; and, B and C greater than A. It is required to make a triangle of which the sides shall be equal to A, B, C, each to each.



Take a straight line D E terminated at the point D, but unlimited towards E. Make (I. 3) DF equal to A, FG equal to B, and GH equal to C. From the centre F, at the distance FD, describe (*Post.* 3) the circle DKL. From the centre G, at the distance GH, describe (*Post.* 3) another circle HLK. And join KF, KG. The triangle KFG has its sides equal to the three straight lines A, B, C.

Because the point F is the centre of the circle DKL, FD is equal (*Def.* 15) to FK. But FD is equal (*Const.*) to the straight line A. Therefore FK is equal (*Ax.* 1) to A. Again, because G is the centre of the circle LKH, GH is equal (*Def.* 15) to GK. But GH is equal to C. Therefore also GK is equal to C. And FG is equal (*Const.*) to B. Therefore the three straight lines KF, FG, GK, are equal to the three straight lines A, B, C. Wherefore the triangle KFG has been made, having its three sides KF, FG, GK, equal to the three given straight lines A, B, C. Q. E. F.

This is the general proposition of which Prop. I. is but a particular case. It is evident that upon the other side of the base FG, another triangle might be constructed, having its three sides equal to the three given straight lines.

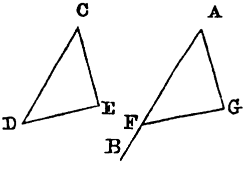
In the demonstration, it is assumed that the two circles will intersect each other. To prove this, it is sufficient to observe that the sum of the radii of the two circles is, by hypothesis, greater than the distance between their centres.

Exercise 1.—**To make a triangle equal to a given triangle.**

Exercise 2.—**To make a rectilineal figure equal to a given rectilineal figure.**

PROP. XXIII. PROBLEM.

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.



Let AB be the given straight line, A the given point in it, and DCE the given rectilineal angle. It is required to make an angle at the given point A in the given straight line AB, that shall be equal to the given rectilineal angle DCE.

In CD, CE, take any points D, E, and join DE. Upon the straight line AB make (I. 22) the triangle AFG, the sides of which shall be equal to the three straight lines CD, DE, EC, that is, AF equal to CD, AG to CE, and FG to DE. The angle FAG is equal to the angle DCE.

Because the two sides FA, AG, are equal to the two sides DC, CE, each to each, and the base FG to the base DE. Therefore the angle FAG is equal (I. 8) to the angle DCE. Wherefore at the given point A, in the given straight line AB, the angle FAG is made equal to given rectilineal angle DCE. Q. E. F.

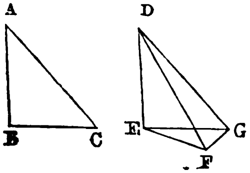
It is evident that upon the other side of the straight line AB, another angle might be made equal to the given angle DCE.

Exercise.—At a given point in a given straight line, to make an angle equal to the supplement of a given angle; also, to make an angle equal to its complement.

PROP. XXIV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them, of the other; the base of that which has the greater angle is greater than the base of the other.

Let ABC, DEF be two triangles, which have the two sides AB, AC, equal to the two sides DE, DF, each to each; viz., AB equal to DE, and AC to DF. But the angle BAC greater than the angle EDF. The base BC is greater than the base EF.



Of the two sides DE, DF, let DE be the side which is not greater than the other. At the point D, in the straight line DE, make (I. 23) the angle EDG equal to the angle BAC. Make DG equal (I. 3) to AC or DF. And join EG, GF.

Because DE is equal (*Hyp.*) to AB, and DG (*Const.*) to AC, the two sides, ED, DG, are equal to the two BA, AC, each to each. And the angle EDG is equal (*Const.*) to the angle BAC. Therefore the base EG is equal (I. 4) to the base BC. Again, because DG is equal to DF, the angle DFG is equal (I. 5) to the angle DGF. But the angle DGF is greater (*Ax.* 9) than the angle EGF. Therefore the angle DFG is also greater than EGF. Much more then is the angle EFG greater than the angle EGF. Now, because the angle EFG of the triangle EFG is greater than its angle EGF, and the greater (I. 19) angle is subtended by the greater side. Therefore the side EG is greater than the side EF. But EG was proved to be equal to BC. Therefore BC is greater than EF. Therefore if two triangles, &c. Q. E. D.

Dr. Simson in the construction of this proposition, introduced these words: "of the two sides DE, DF, let DE be the side which is not greater than the other," in order to avoid three distinct cases of construction, which would arise by taking that side *which is greater* than the other.

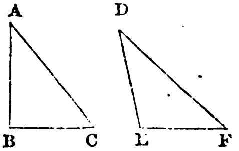
Exercise 1.—Demonstrate this proposition, by making the construction on the greater of the two sides of the triangle DEF, and exhibit the three distinct cases above mentioned.

Exercise 2.—Demonstrate that in Dr. Simson's construction, the straight line EG cuts the straight line DF in some point between D and F.

PROP. XXV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other; the angle contained by the two sides of that which has the greater base, is greater than the angle contained by the two sides equal to them of the other.

Let ABC, DEF be two triangles which have the two sides AB, AC equal to the two sides DE, DF, each to each; viz., AB equal to DE and AC to DF. But the base BC greater than the base EF. The angle BAC is greater than the angle EDF.



For, if the angle BAC be not greater than the angle EDF, it must either be equal to, or less than the angle EDF. The angle BAC is not equal to the angle EDF, because then the base BC would be equal (I. 4) to the base EF: but it is (*Hyp.*) not equal. Therefore the angle BAC is not equal to the angle EDF. Again, the angle BAC is not less than the angle EDF, because then the base BC would be less (I. 24) than the base EF: but it is (*Hyp.*) not less. Therefore the angle BAC is not less than the angle EDF. And it was shown that the angle BAC is not equal to the angle EDF. Therefore the angle BAC is greater than the angle EDF. Wherefore, if two triangles, &c. Q. E. D.

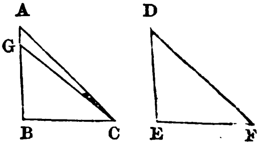
Corollary. **If two triangles have two sides of the one respectively equal to two sides of the other, the base of the one is greater than, equal to, or less than the base of the other, according as the angle opposite to the base of the one is greater than, equal to, or less than the angle opposite to the base of the other.**

PROP. XXVI. THEOREM.

If two triangles have two angles of the one equal to two angles of the other, each to each; and one side equal to one side,—viz., either the sides adjacent to the equal angles, or the sides opposite to equal angles in each; then their other sides are equal, each to each, and also the third angle of the one to the third angle of the other.

Let ABC, DEF be two triangles which have the two angles ABC, BCA of the one, equal to the two angles DEF, EFD of the other, each to each; viz., ABC to DEF, and BCA to EFD. Also, a side of the one triangle equal to a side of the other.

First, let those sides be equal which are adjacent to the angles that are equal in the two triangles; viz., BC to EF. Then their other sides are equal, each to each; viz., AB to DE, and AC to DF; and the third angle BAC is equal to the third angle EDF.

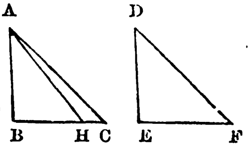


For, if AB be not equal to DE, one of them must be greater than the other. Let AB be the greater of the two. Make BG equal (I. 3) to DE, and join GC.

Because in the two triangles GBC, DEF, BG is equal (*Const.*) to DE, and BC (*Hyp.*) to EF, the two sides GB, BC are equal to the two sides DE, EF, each to each. But the angle GBC is equal (*Hyp.*) to the angle DEF. Therefore the base GC is equal (I. 4) to the base DF, and the triangle GBC to the triangle DEF. And the remaining angles of the one are equal to the remaining angles of the other, each to each; viz., those to which the equal sides are opposite. Therefore the angle GCB is equal to the angle DFE. But the angle DFE is (*Hyp.*) equal to the angle BCA. Wherefore also the angle BCG is equal (*Ax.* 1) to the angle BCA, the less to the greater, which is impossible. Therefore the side AB is not unequal to the side DE; that is, AB is equal to DE.

And BC is equal (*Hyp.*) to EF. Therefore the two sides AB, BC are equal to the two sides DE, EF, each to each. And the angle ABC is equal (*Hyp.*) to the angle DEF. Therefore the base AC is equal (I. 4) to the base DF, and the third angle BAC to the third angle EDF.

Next, let those sides which are opposite to equal angles in each triangle be equal to one another; viz., AB to DE. Then their other sides are equal; viz., AC to DF, and BC to EF. And the third angle BAC is equal to the third angle EDF.



For, if BC be not equal to EF, one of them must be greater than the other. Let BC be the greater of the two. Make BH equal (I. 3) to EF, and join AH.

Because in the two triangles ABH, DEF, BH is equal (*Const.*) to EF, and AB to (*Hyp.*) DE; the two sides AB, BH are equal to the two sides DE, EF, each to each. But the angle ABH is equal (*Hyp.*) to the angle DEF. Therefore the base AH is equal to the base DF, and the triangle ABH to the triangle DEF. And the remaining angles of the one are equal to the remaining angles of the other, each to each; viz., those to which the equal sides are opposite. Therefore the angle BHA is equal to the angle EFD. But the angle EFD is equal (*Hyp.*) to the angle BCA. Therefore also the angle BHA is equal (*Ax.* 1) to the angle BCA; that is, the exterior angle BHA of the triangle AHC is equal to its interior and opposite angle BCA; which is impossible (I. 16). Therefore BC is not unequal to EF; that is, BC is equal to EF. And AB is equal (*Hyp.*) to DE. Therefore the two sides AB, BC are equal to the two sides DE, EF, each to each. And the angle ABC is equal (*Hyp.*) to the angle DEF. Therefore the base AC is equal (I. 4) to the base DF, and the third angle BAC to the third angle EDF. Therefore, if two triangles, &c. Q. E. D.

The enunciation of this proposition may be thus simplified: If two triangles have two angles of the one, equal to two angles of the other, each to each, and a side of the one equal to a side of the other similarly situated as to the equal angles, the two triangles are equal in every respect.

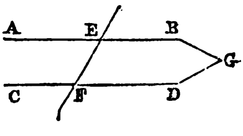
***Exercise* 1.—In an isosceles triangle, if a straight line be drawn from the angle opposite the base, bisecting the angle, it bisects the base; or, if it bisect the base, it bisects the angle; and in either case, it cuts the base at right angles.**

***Exercise* 2.—Through a given point to draw a straight line which shall make equal angles with two straight lines given in position.**

PROP. XXVII. THEOREM.

If a straight line falling on two other straight lines, make the alternate angles equal to each other; these two straight lines are parallel.

Let the straight line EF, which falls upon the two straight lines AB, CD, make the alternate angles AEF, EFD, equal to one another. Then AB shall be parallel to CD.



For, if AB be not parallel to CD, AB and CD being produced will meet either towards A and C, or towards B and D. Let AB, CD be produced and meet towards B and D, in the point G. Then GEF is a triangle, and its exterior angle AEF is greater (I. 16) than its interior and opposite angle EFG. But the angle AEF is equal *(Hyp.)* to the angle EFG. Therefore the angle AEF is both greater than, and equal to the angle EFG; which is impossible. Wherefore AB, CD being produced, do not meet towards B, D. In like manner, it may be proved, that they do not meet when produced towards A, C. But those straight lines in the same plane, which do not meet either way, though produced ever so far, are parallel *(Def. 35)* to one another. Therefore AB is parallel to CD. Wherefore, if a straight line, &c. Q. E. D.

The angles AEF, EFD, are called *alternate angles*, or more properly, *interior alternate angles*, because they are on opposite sides of the straight line EF, and the one has its vertex at E the one extremity of the portion *between* the parallels, while the other has its vertex at F the other extremity of the same.

In the diagram, the crooked lines EBG, FDG, must be considered straight lines, and the figure EFDGB, a triangle, for the sake of the argument.

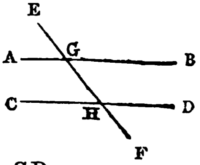
Exercise 1.—If a straight line falling upon two other straight lines, make the *exterior alternate* angles AGE, FHD (see fig. to next proposition) equal to each other, these two straight lines are parallel.

Exercise 2.—If a straight line falling upon two other straight lines, make the two *exterior angles* on the same side of it, equal to two right angles, these two straight lines are parallel.

PROP. XXVIII. THEOREM.

If a straight line falling upon two other straight lines, make the exterior angle equal to the interior and opposite angle upon the same side of the straight line; or make the two interior angles upon the same side of it, together equal to two right angles; these two straight lines are parallel to one another.

Let the straight line EF, falling upon the two straight lines AB, CD, make the exterior angle EGB equal to the interior and opposite angle GHD upon the same side of EF; or make the two interior angles BGH, GHD on the same side of it, together equal to two right angles. Then AB is parallel to CD.



Because the angle EGB is equal (*Hyp.*) to the angle GHD. And the angle EGB is equal (I. 15) to the angle AGH. Therefore the angle AGH is equal (*Ax.* 1) to the angle GHD; and they are alternate angles. Therefore AB is parallel (I. 27) to CD.

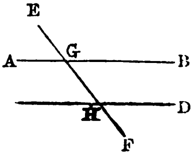
Again, because the two angles BGH, GHD are together equal (*Hyp.*) to two right angles. And the two angles AGH, BGH are also together equal (I. 13) to two right angles. Therefore the two angles AGH, BGH are equal (*Ax.* 1) to the two angles BGH, GHD. Take away from these equals, the common angle BGH. Therefore the remaining angle AGH is equal (*Ax.* 3) to the remaining angle GHD; and they are alternate angles. Therefore AB is parallel (I. 27) to CD. Wherefore, if a straight line, &c. Q. E. D.

The twelfth axiom will now be admitted as a corollary to this proposition; especially when Prop. XVII. and the note added to that axiom are taken into account.

PROP. XXIX. THEOREM.

If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another; the exterior angle equal to the interior and opposite angle upon the same side of the straight line; and the two interior angles upon the same side of it together equal to two right angles.

Let the straight line EF fall upon the two parallel straight lines AB, CD. The two alternate angles AGH, GHD are equal to one another. The exterior angle EGB is equal to the interior and opposite angle GHD upon the same side of the straight line EF. And the two interior angles BGH, GHD upon the same side of it are together equal to two right angles.



For, if the alternate angles AGH, GHD be not equal, one of them must be greater than the other. Let the angle AGH be greater than the angle GHD. To each of these unequals, add the angle BGH. Then the two angles AGH, BGH, are greater (*Ax.* 4) than the two angles BGH, GHD. But the two angles AGH, BGH, are equal (I. 13) to two right angles. Therefore the two angles BGH, GHD, are less than two right angles. But those straight lines, which with another straight line falling upon them, make the two interior angles on the same side less than two right angles, will meet together (*Ax.* 12) if continually produced. Therefore the two straight lines AB, CD, if produced far enough, will meet. But they never meet, since (*Hyp.*) they are parallel. Therefore the angle AGH is not unequal to the angle GHD; that is, the angle AGH is equal to the angle GHD.

Again, the angle AGH is equal (I. 15) to the angle EGB. Therefore the angle EGB is equal (*Ax.* 1) to the angle GHD.

Lastly, to each of these equals, add the angle BGH. Then the two angles EGB, BGH, are equal (*Ax.* 2) to the two angles BGH, GHD. But the two angles EGB, BGH are equal (I. 13) to two right angles. Therefore also the two angles BGH, GHD are equal (*Ax.* 1) to two right angles. Wherefore, if a straight line, &c. Q. E. D.

Corollary 1.—If two angles have their legs parallel each to each, and proceeding from their vertices in the same directions, they are equal to each other.

Corollary 2.—If two angles be equal to each other, and a leg of the one be parallel to a leg of the other, their remaining legs are parallel.

Exercise.—If a straight line be drawn perpendicular to one of two parallel straight lines, it is also perpendicular to the other.

PROP. XXX. THEOREM.

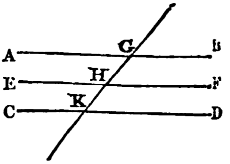
Straight lines which are parallel to the same straight line are parallel to each other.

Let the straight lines AB, CD, be each of them parallel to EF. Then AB is also parallel to CD.

Draw the straight line GHK cutting the three straight lines AB, EF, and CD.

Because the straight line GHK cuts the parallel straight lines AB, EF, the angle AGH is equal (I. 29) to the alternate angle GHF

Again, because the straight line GHK cuts the parallel straight lines EF, CD, the exterior angle GHF is equal (I. 29) to the interior angle HKD. But it was proved that the angle AGH is equal to the angle GHF. Therefore the angle AGH is equal (*Ax.* 1) to the angle GKD; and these are alternate angles. Therefore AB is parallel (I. 27) to CD. Wherefore, straight lines which are, &c. Q. E. D.



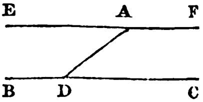
The student should prove the case of this proposition, when the straight line EF is not between the straight lines AB, CD, but on either side of both.

PROP. XXXI. PROBLEM.

To draw through a given point a straight line parallel to a given straight line.

Let A be the given point, and BC the given straight line. It is required to draw, through the point A, a straight line parallel to the straight line BC.

In the straight line BC take any point D, and join AD. At the point A in the straight line AD, make (I. 23) the angle DAE equal to the angle ADC. Produce the straight line EA to F. Then EF is parallel to BC.



Because the straight line AD meets the two straight lines EF, BC, and makes the alternate angles EAD, ADC, equal to one another. Therefore EF is parallel (I. 27) to BC. Wherefore, through the given point A, a straight line EAF has been drawn parallel to the given straight line BC. Q. E. F.

The application of the 23rd Proposition of this book, is unnecessary, if the 11th and 12th Propositions be employed. In the construction and demonstration, either of the cases of the 28th Proposition may be used instead of the 27th Proposition.

***Exercise.*—Of all triangles having the same vertical angle, and having their bases passing through the same point, the least is that whose base is bisected in that point.**

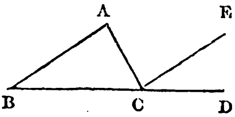
PROP. XXXII. THEOREM.

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.

Let ABC be a triangle, and let one of its sides BC be produced to D. Then the exterior angle ACD is equal to the two interior and opposite angles CAB, ABC. And the three interior angles ABC, BCA, CAB are equal to two right angles.

Through the point C draw the straight line (I. 31) CE parallel to the side BA.

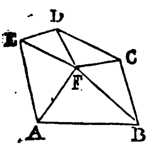
Because CE is parallel to BA, and AC meets them, the angle ACE is equal (I. 29) to the alternate angle BAC. Again, because CE is parallel to AB and BD



falls upon them, the exterior angle E C D is equal (I. 29) to the interior and opposite angle A B C. But the angle A C E was shown to be equal to the angle B A C. Therefore the whole exterior angle A C D is equal (*Ax.* 2) to the two interior and opposite angles C A B, A B C. To each of these equals, add the angle A C B. Therefore the two angles A C D, A C B are equal (*Ax.* 2) to the three angles C A B, A B C, A C B. But the two angles A C D, A C B are equal (I. 13) to two right angles. Therefore also the three angles C A B, A B C, A C B are equal (*Ax.* 1) to two right angles. Wherefore, if a side of any triangle be produced, &c. Q. E. D.

Cor. 1.—All the interior angles of any rectilineal figure together with four right angles are equal to twice as many right angles as the figure has sides.

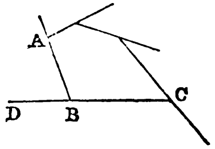
Let A B C D E be any rectilineal figure. All the interior angles A B C, B C D, &c. together with four right angles are equal to twice as many right angles as the figure has sides.



Divide the rectilineal figure A B C D E into as many triangles as the figure has sides, by drawing straight lines from a point F within the figure to each of its angles.

Because the three interior angles of a triangle are equal (I. 32) to two right angles, and there are as many triangles in the figure as it has sides, all the angles of these triangles are equal to twice as many right angles as the figure has sides. But all the angles of these triangles are equal to the interior angles of the figure, viz. A B C, B C D, &c., together with the angles at the point F, which are equal (I. 15. *Cor.* 2) to four right angles. Therefore all the angles of these triangles are equal (*Ax.* 1) to the interior angles of the figure together with four right angles. But it has been proved that all the angles of these triangles are equal to twice as many right angles as the figure has sides. Therefore all the angles of the figure together with four right angles are equal to twice as many right angles as the figure has sides.

Cor. 2.—All the exterior angles of any rectilineal figure, made by producing the sides successively in the same direction, are together equal to four right angles.



Because the interior angle A B C, and its adjacent exterior angle A B D, are (I. 13) together equal to two right angles. Therefore all the interior angles, together with all the exterior angles of the figure, are equal to twice as many right angles as the figure has sides. But it has been proved by the foregoing corollary, that all the interior angles together with four right angles are equal to twice as many right angles as the figure has sides. Therefore all the interior angles together with all the exterior angles are equal (*Ax.* 1) to all the interior angles and four right angles. Take from these equals all the interior angles. Therefore all the exterior angles of the figure are equal (*Ax.* 3) to four right angles.

Cor. 1—If two angles of a triangle be given, the third is given; for, it is the difference between their sum and two right angles.

Cor. 2.—If two angles of one triangle be equal to two angles of another triangle, the third angle of the one is equal to the third angle of the other.

Cor. 3.—Every angle of an equilateral triangle is equal to one-third of two right angles, or two-thirds of a right angle. Hence, a right angle can be *trisected.*

Cor. 4.—If one angle of a triangle be a right angle, the sum of the other two is a right angle.

Cor. 5.—If one angle of a triangle be equal to the sum of the other two, it is a right angle.

Cor. 6.—If one angle of a triangle be greater than the sum of the other two, it is obtuse; and if less, acute.

Cor. 7.—In every isosceles right-angled triangle, each of the acute angles is equal to half a right angle.

Cor. 8.—All the interior angles of every quadrilateral figure are together equal to four right angles. This is only a particular case of Euclid's Cor. 1; but it is very necessary to be remembered.

Exercise 1.—If a straight line be drawn from one of the angles of a triangle, making the exterior angle equal to the two interior and opposite angles, it is in the same straight line with the adjacent side.

Exercise 2.—To *trisect* a given finite straight line; that is, to divide it into three equal parts.

Exercise 3.—Any angle of a triangle is right, acute or obtuse, according as the straight line drawn from its vertex bisecting the opposite side, is equal to, greater than, or less than half that side.

Exercise 4.—The straight line drawn from the vertex of any angle of a triangle bisecting the opposite side, is equal to, greater than, or less than half that side, according as the angle is right, acute or obtuse.

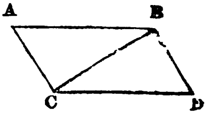
Exercise 5.—If the sides of an equilateral and equiangular *pentagon* or five-sided figure, be produced till they meet, the angles formed at the points of meeting, are together equal to two right angles.

Exercise 6.—If the sides of an equilateral and equiangular *hexagon*, or six-sided figure, be produced till they meet, the angles formed at the points of meeting, are together equal to four right angles.

PROP. XXXIII. THEOREM.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are also themselves equal and parallel.

Let the straight lines A C, B D join the two equal and parallel straight lines A B, C D towards the same parts. Then the straight lines A C, B D are also equal and parallel.



Join B C. Because A B is parallel to C D, and B C meets them, the angle A B C is equal (I. 29) to the alternate angle B C D. Because A B is equal to C D, and B C common to the two triangles A B C, D C B; the two sides A B, B C, are equal to the two D C, C B, each to each. And the angle A B C was proved to be equal to the angle B C D. Therefore the base A C is equal (I. 4) to the base B D, and the triangle A B C to the triangle B C D. Also, the remaining angles of the one are equal to the remaining angles of the other, each to each; viz., those to which the equal sides are opposite. Therefore the angle A C B is equal to the angle C B D. Because the straight line B C meets the two straight lines A C, B D, *and makes the alternate* angles A C B, C B D equal to one another.

Therefore AC is (I. 27) parallel to BD; and AC was proved to be equal to BD. Therefore, the straight lines which, &c. Q. E. D.

The enunciation of this proposition is more clearly expressed thus: "The straight lines which, *without crossing each other*, join the extremities of two equal and parallel straight lines, are themselves equal and parallel."

Corollary.—A quadrilateral which has two of its opposite sides equal and parallel, is a parallelogram.—See the following definition:—

DEFINITION XXXVI.

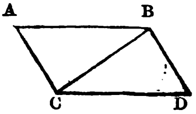
A parallelogram is a four-sided figure of which the opposite sides are parallel; and the diagonal is the straight line joining the vertices of two opposite angles.

PROP. XXXIV. THEOREM.

The opposite sides and angles of a parallelogram are equal to one another, and the diagonal bisects it, that is, divides it into two equal parts.

Let AD be a parallelogram, of which BC is a diagonal. The opposite sides and angles of the figure are equal to one another; and the diagonal BC bisects it.

Because AB is parallel to CD, and BC meets them, the angle ABC is equal (I. 29) to the alternate angle BCD. Because AC is parallel to to BD, and BC meets them, the angle ACB is equal (I. 29) to the alternate angle CBD. Because in the two triangles ABC, CBD, the two angles ABC, BCA, in the one, are equal to the two angles BCD, CBD in the other, each to each; and one side BC, adjacent to these equal angles, is common to the two triangles. Therefore their other sides are equal, each to each, and the third angle of the one is equal to the third angle of the other (I. 26); viz., the side AB to the side CD, the side AC to the side BD, and the angle BAC to the angle BDC. Because the angle ABC is equal to the angle BCD, and the angle CBD to the angle ACB. Therefore the whole angle ABD is equal (*Ax.* 2) to the whole angle ACD. And the angle BAC has been proved to be equal to the angle BDC. Therefore the opposite sides and angles of a parallelogram are equal to one another.



Also the diagonal BC bisects the parallelogram AD. Because in the two triangles ABC, DCB, AB is equal to CD, and BC common, the two sides AB, BC, are equal to the two sides DC, CB, each to each. And the angle ABC has been proved to be equal to the angle BCD. Therefore the triangle ABC is equal (I. 4) to the triangle BCD. Wherefore the diagonal BC divides the parallelogram AD into two equal parts. Q. E. D.

Corollary 1.—If a parallelogram have one angle a right angle, all its angles are right angles.

Corollary 2.—Parallelograms, having one angle equal in each, are equiangular.

Corollary 3.—Parallelograms, which have one angle and two adjacent sides equal, in each, are equal in all respects.

Corollary 4.—The adjacent angles of a parallelogram are supplements of each other.

Exercise 1.—If the opposite sides, or the opposite angles of a quadrilateral figure be equal, it is a parallelogram.

Exercise 2.—The diagonals of a parallelogram bisect each other; and if the diagonals of a quadrilateral bisect each other, it is a parallelogram.

Exercise 3.—The diagonals of rectangular parallelograms are equal; and in oblique-angled parallelograms, those which join the vertices of the acute angles are greater than those which join the obtuse.

Exercise 4.—To divide a straight line into any number of equal parts.

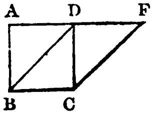
Exercise 5.—To bisect a parallelogram by a straight line drawn through any point in one of its sides.

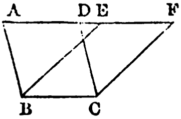
N.B. In naming a parallelogram by letters, it is customary, and it is sufficient, to take the two letters only which are at the vertices of any two of its opposite angles.

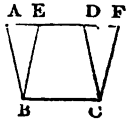
PROP. XXXV. THEOREM.

Parallelograms upon the same base, and between the same parallels, are equal to one another.

Let the parallelograms AC, BF be upon the same base BC, and and between the same parallels AF, BC. The parallelogram AC is equal to the parallelogram BF.







First, let the sides AD, DF of the parallelograms AC, BF, opposite to the base BC, be terminated in the same point D.

Because each of the parallelograms AC, BF, is double (I. 34) of the triangle BDC. Therefore the parallelogram AC is equal (*Ax.* 6) to the parallelogram BF.

Next, let the sides AD, EF, opposite to the base BC, be not terminated in the same point.

Because AC is a parallelogram, AD is equal (I. 34) to BC. For a similar reason, EF is equal to BC. Therefore AD is equal (*Ax.* 1) to EF; and DE is common to both. Wherefore the whole, or the remainder AE, is equal to the whole, or the remainder DF (*Ax.* 2 or 3); and AB is equal (I. 34) to DC. Because in the triangles EAB, FDC, the side FD is equal to the side EA, and the side DC to the side AB, and the exterior angle FDC is equal (I. 29) to the interior and opposite angle EAB. Therefore the base FC is equal (I. 4) to the base EB, and the triangle FDC to the triangle EAB. From the trapezium ABCF, take the triangle FDC, and the remainder is the parallelogram AC. From the same trapezium take the triangle EAB and the remainder is the parallelogram BF. But when equals are taken from equals, or from the same, the remainders (*Ax.* 3) are equal. Therefore the parallelogram AC is equal to the parallelogram BF. Therefore, parallelograms upon the same, &c. Q. E. D.

In Dr. Thomson's edition, this highly important proposition is simplified by the application of Prop. XXVI. of this book. It may be simplified still more in the following *manner*:—Because AB is equal to CD, and BE to CF (I. 34), *in the two triangles* ABE, DCF, the two sides AB, BE, are equal to the two sides DC, CF. And the angle ABE is equal to the angle DCF (I. 29 *Cor.* 1). Therefore, the triangle ABE is equal (I. 4) to the triangle DCF. This equality being proved, the rest of the demonstration is the same as that in the text. That part indeed is often rendered obscure by reference to Axiom 1., instead of a new one, tacitly assumed by Euclid; viz., that "if equals be taken from the *same thing*, the remainder are equal."

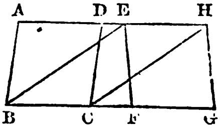
This proposition is the foundation of the mensuration of plane surfaces, and hence of land-measuring. As the area of a rectangle is determined practically by multiplying its length by its breadth, or its base by its altitude, and as by this proposition, every parallelogram having the same base and altitude (that is, the same perpendicular breadth between the parallels) with a rectangle, is equal to that rectangle in area; therefore the area of every parallelogram is found by multiplying the length of its base, by its altitude.

Exercise. Equal parallelograms upon the same base and on the same side of it, are between the same parallels.

PROP. XXXVI. THEOREM.

Parallelograms upon equal bases and between the same parallels, are equal to one another.

Let AC, EG be parallelograms upon equal bases BC, FG, and between the same parallels AH, BG. The parallelogram AC is equal to the parallelogram EG.



Join BE, CH. Because BC is equal to FG, (*Hyp.*) and FG to EH (I. 34), BC is equal to EH (*Ax.* 1). But BC and EH, are parallel, and joined towards the same parts by the straight lines BE, CH. And straight lines which join the extremities of equal and parallel straight lines towards the same parts, are (I. 33) themselves equal and parallel. Therefore the straight lines BE, CH are both equal and parallel. Wherefore BH is a parallelogram (*Def.* 36). Because the parallelograms AC, BH, are upon the same base BC, and between the same parallels BC, AH, the parallelogram AC is equal (I. 35) to the parallelogram BH. Because the parallelograms GE, HB are upon the same base EH, and between the same parallels GB, HE, the parallelogram EG is equal to the parallelogram BH. Therefore the parallelogram AC is equal (*Ax.* 1) to the parallelogram EG. Therefore, parallelograms upon equal bases, &c. Q. E. D.

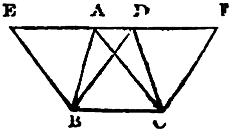
Exercise 1.—If the base of a parallelogram be equal to half the sum of the two parallel sides of a trapezoid, between the same parallels, the parallelogram is equal to the trapezoid.

Exercise 2.—Demonstrate the Theorem of Pappus: The parallelograms described on any two sides of a triangle, are together equal to the parallelogram described on the base, having its side equal and parallel to the straight line drawn from the point of intersection of the exterior sides of the former, to the vertex of the triangle.

PROP. XXXVII. THEOREM.

Triangles upon the same base, and between the same parallels, are equal to one another.

Let the triangles ABC, DBC be upon the same base BC, and between the same parallels AD, BC. The triangle ABC is equal to the triangle DBC.



Produce AD both ways to the points E and F. Through B draw BE parallel to CA (I. 31), and through C draw CF parallel to BD. Then each of the figures E C, B F, is a parallelogram (*Def.* 36.)

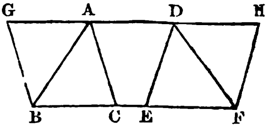
The parallelograms EC, BF, are equal (I. 35), because they are upon the same base BC, and between the same parallels BC, EF. The triangle A B C is half of the parallelogram E C (I. 34), because the diagonal AB bisects it. Also, the triangle D B C is half of the parallelogram B F, because the diagonal DC bisects it. But the halves of equal things are equal (*Ax.* 7). Therefore the triangle A B C is equal to the triangle D B C. Wherefore, triangles, &c. Q. E. D.

Exercise. To describe a triangle equal to any given rectilineal figure.

In solving this problem, the learner may begin with a parallelogram; then proceed to a trapezium, a pentagon, a hexagon, and so on. He will then ultimately find that, if a figure had a thousand sides, he could reduce it, by degrees, to an equivalent triangle.

PROP. XXXVIII. THEOREM.

Triangles upon equal bases, and between the same parallels, are equal to one another.



Let the triangles A B C, D E F, be upon equal bases B C, E F, and between the same parallels B F, A D. The triangle A B C is equal to the triangle D E F.

Produce A D both ways to the points G and H. Through B draw B G parallel to C A (I. 31), and through F draw F H parallel to E D. Then each of the figures G C, E H, is a parallelogram.

The parallelograms G C, E H are equal to one another (I. 36), because they are upon equal bases B C, E F, and between the same parallels B F, G H. The triangle A B C is the half of the parallelogram G C (I. 34), because the diagonal A B bisects it. And the triangle D E F is the half of the parallelogram E H, because the diagonal D F bisects it. But the halves of equal things are equal (*Ax.* 7). Therefore the triangle A B C is equal to the triangle D E F. Wherefore, triangles upon equal bases, &c. Q. E. D.

Corollary 1.—The straight line drawn from any angle of a triangle bisecting the opposite side, bisects the triangle.

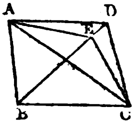
Corollary 2.—If two triangles have two sides of the one equal to two sides of the other, each to each, and the angle contained by the two sides of the one, the supplement of the angle contained by the two sides of the other, these triangles are equal.

Exercise 1.—To bisect a triangle by drawing a straight line through any point in one of its sides.

Exercise 2.—The straight lines drawn from the opposite angles of a parallelogram to any point in a diagonal, divide it into equal triangles on opposite sides of the segments of the diagonal.

PROP. XXXIX. THEOREM.

Equal triangles upon the same base, and upon the same side of it, are between the same parallels.



Let the equal triangles A B C, D B C, be upon the same base B C, and upon the same side of it. The triangles A B C, D B C, are between the same parallels; that is, if A B be joined, A D is parallel to B C.

For if A D be not parallel to D C, through the point A draw A E parallel to B C (I. 31), meeting B D in E; and join E C.

The triangle A B C is equal to the triangle E B C (I. 37), because they are upon the same base B C, and between the same parallels B C, A E. But the triangle A B C is equal (*Hyp.*) to the triangle D B C. Therefore the triangle D B C is equal to the triangle E B C, the greater to the less, which is impossible. Therefore A E is not parallel to B C. In the same manner it can be proved that no other straight line but A D is parallel to B C. Therefore A D is parallel to B C. Wherefore, equal triangles upon, &c. Q. E. D.

Exercise 1.—If two sides of a triangle be bisected, the straight line joining the points of bisection is parallel to the base, and equal to half of it.

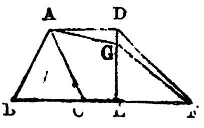
Exercise 2.—If the sides of any quadrilateral figure be bisected, and the points of bisection be joined, the figure thus formed will be a parallelogram, and equal to half the quadrilateral.

PROP. XL. THEOREM.

Equal triangles upon equal bases in the same straight line, and on the same side of it, are between the same parallels.

Let the equal triangles A B C, D E F, be upon equal bases B C, E F, in the same straight line B F, and on the same side of it; they are between the same parallels; that is, if A D be joined, A D is parallel to B F.

For, if A D be not parallel to B F, through A draw A G parallel to B F (I. 31), meeting E D in G, and join G F.

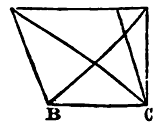


The triangle A B C is equal to the triangle G E F (I. 38), because they are upon equal bases B C, E F, and between the same parallels B F, A G. But the triangle A B C is equal (*Hyp.*) to the triangle D E F. Therefore the triangle D E F is equal (*Ax.* 1) to the triangle G E F, the greater equal to the less, which is impossible. Therefore A G is not parallel to B F. In the same manner it can be proved that no other straight line is parallel to B F, but A D. Therefore A D is parallel to B F. Wherefore, equal triangles upon, &c. Q. E. D.

In this proposition and the preceding one, the demonstration would be conducted in the same manner, if the parallels A E and A G were to meet B D and E D produced. The argument would then rest on the absurdity of the less triangle being equal to the greater.

PROP. XLI. THEOREM.

If a parallelogram and a triangle be upon the same base, and between the same parallels; the parallelogram is double of the triangle.



Let the parallelogram B D, and the triangle E B C be upon the same base B C, and between the same parallels B C, A E. The parallelogram B D is double of the triangle E B C.

Join A C. The triangle A B C is equal to the triangle E B C (I. 37), because they are upon the same base B C, and between the same parallels B C, A E. But the parallelogram B D is double of the triangle A B C (I. 34), because the diagonal A C bisects it. Therefore the parallelogram B D is also double of the triangle E B C. Therefore, if a parallelogram and a triangle, &c. Q. E. D.

This proposition is the foundation of the mensuration of triangles, and consequently of all rectilineal figures, as they can easily be divided into triangles. Inasmuch as the area of a parallelogram is found by multiplying its base by its perpendicular altitude or breadth, so the area of a triangle is found by multiplying its base by its perpendicular altitude, and taking half the product. In a triangle the perpendicular altitude is the perpendicular drawn from the vertex of the angle opposite to the base, to the base itself or to the base produced.

Corollary.—A parallelogram is equal to a triangle on the double of its base, and between the same parallels.

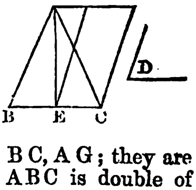
Exercise 1.—If from any point within a parallelogram, straight lines be drawn to the extremities of two opposite sides, the two triangles upon these sides are together equal to half of the parallelogram.

Exercise 2.—In a trapezoid, if one of the sides which is not parallel be bisected and straight lines be drawn from the point of bisection to the extremities of the other side which is not parallel, the triangle which they form with the latter side is equal to half the trapezoid.

PROP. XLII. PROBLEM.

To describe a parallelogram equal to a given triangle, and having one of its angles equal to a given rectilineal angle.

Let A B C be the given triangle, and D the given rectilineal angle. It is required to describe a parallelogram equal to the given triangle A B C, and having one of its angles equal to D.



Bisect B C in E (I. 10), and join A E. At the point E in the straight line E C, make the angle C E F (I. 23) equal to the angle D. Through A draw A F G (I. 31), parallel to B C, and through C draw C G parallel to E F. Then the figure C E F G is a parallelogram (*Def.* 36).

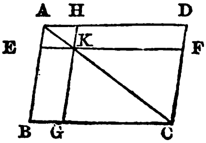
Because the two triangles A B E, A E C are on equal bases B E, E C, and between the same parallels B C, A G; they are equal (I. 38) to one another. Therefore the triangle A B C is double of the triangle A E C. But the parallelogram F C is double of the triangle A E C (I. 41), because they are upon the same base E C, and between the same parallels E C, A G. Therefore the parallelogram

F C is equal (*Ax.* 6) to the triangle A B C, and it has one of its angles C E F equal to the given angle D. Wherefore, a parallelogram F C has been described equal to the given triangle A B C, and having one of its angles C E F equal to the given angle D. Q. E. D.

Exercise.—To describe a triangle equal to a given parallelogram, and having an angle equal to a given rectilineal angle.

PROP. XLIII. THEOREM.

The complements of the parallelograms, which are about the diagonal of any parallelogram, are equal.



Let B D be a parallelogram, of which the diagonal is A C; and E H, G F the parallelograms about A C, that is, through which A C passes. Then B K, K D are the other parallelograms which make up the whole figure B D, and are therefore called the complements. The complement B K is equal to the complement K D.

Because B D is a parallelogram, and A C its diagonal, the triangle A B C is equal (I. 34) to the triangle A D C. Again, because E H is a parallelogram, and A K its diagonal, the triangle A E K is equal (I. 34) to the triangle A H K. For the same reason, the triangle K G C is equal to the triangle K F C. Therefore the two triangles A E K, K G C are equal (*Ax.* 2) to the two triangles A H K, K F C. But the whole triangle A B C is equal to the whole triangle A D C. Therefore the remaining complement B K is equal (*Ax.* 3) to the remaining complement K D. Wherefore the complements, &c. Q. E. D.

Corollary 1.—The parallelograms about the diagonal of a parallelogram, as also its complements, are equiangular to the whole parallelogram and to each other.

Corollary 2.—If through any point in the diagonal of a parallelogram, straight lines be drawn parallel to its sides, the parts into which each divides the parallelogram are equal, the greater to the greater, and the less to the less.

Exercise 1.—In the preceding figure join E H, B D, and G F, and demonstrate that the three diagonals thus drawn, are parallel to each other.

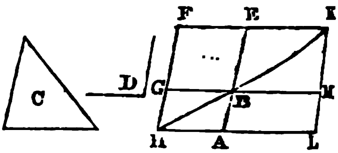
Exercise 2.—If about the diagonal of a parallelogram, any number of parallelograms be placed, whether the extremities of their diagonals coincide in the same point or not, the remaining complementary rectilineal figures, on each side of the diagonal of the parallelogram, are equal.

PROP. XLIV. PROBLEM.

To a given straight line to apply a parallelogram, equal to a given triangle, and having one of its angles equal to a given rectilineal angle.

Let A B be the given straight line, C the given triangle, and D the given rectilineal angle. It is required to apply to (that is, to describe upon) the straight line A B, a parallelogram equal to the triangle C, and having an angle equal to D.

Make the parallelogram B F equal (I. 42) to the triangle C, and having the angle EBG equal to the angle D. Place it so that BE shall be in the same straight line with AB. Produce FG to H. Through A draw AH parallel to BG or EF (I. 31) and meeting FH in H. Join HB.



Because the straight line HF falls upon the two parallels AH, EF, the two angles AHF, HFE are together equal (I. 29) to two right angles. Therefore the two angles BHF, HFE are less than two right angles. But those straight lines which with another straight line, make the two interior angles upon the same side of it less than two right angles, meet (*Ax.* 12) if produced far enough. Therefore HB, FE shall meet, if produced. Let them be produced and meet in K. Through K draw KL parallel to EA or FH, and produce HA, GB, to meet KL in the points L and M.

Because FL is a parallelogram (*Def.* 36), of which the diagonal is HK; AG, ME, are the parallelograms about HK, and LB, BF are the complements. Therefore the complement LB is equal (I. 43) to the complement BF. But the complement BF is equal (*Const.*) to the triangle C. Therefore LB is equal (*Ax.* 1) to the triangle C. Because the angle GBE is equal (I. 15) to the angle ABM, and likewise (*Const.*) to the angle D. Therefore the angle ABM is equal (*Ax.* 1) to the angle D. Wherefore to the straight line AB, is applied the parallelogram LB, equal to the triangle C, and having the angle ABM equal to the angle D. Q. E. F.

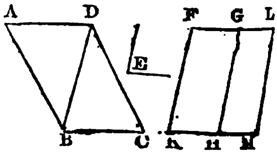
Corollary.—From this proposition, it is manifest how to describe on a given straight line, a rectangle equal to a given triangle.

Exercise.—To a given straight line to apply a triangle equal to a given parallelogram and having an angle equal to a given rectilineal angle.

PROP. XLV. PROBLEM.

To describe a parallelogram equal to a given rectilineal figure and having an angle equal to a given rectilineal angle.

Let ABCD be the given rectilineal figure, and E the given rectilineal angle. It is required to describe a parallelogram equal to the figure ABCD, and having an angle equal to E.



Join DB. Describe the parallelogram FH equal to the triangle ADB, and having the angle FKH equal (I. 42) to the angle E. To the straight line GH apply the parallelogram GM equal to the triangle DBC, and having the angle GHM equal (I. 44) to the angle E. Then the figure KL is the parallelogram required.

Because the angle E is equal (*Const.*) to each of the angles FKH, GHM, the angle FKH is equal (*Ax.* 1) to the angle GHM. To each of these equals, add the angle KHG. Therefore the two angles FKH, KHG *are equal to the* two angles KHG, GHM. But the two angles FKH, *KHG are* equal (I. 29) to two right angles. Therefore the two angles

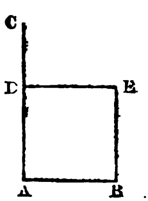
KHG, GHM are also equal (*Ax.* 1) to two right angles. Because at the point H, in the straight line GH, the two straight lines KH, HM, upon opposite sides of it, make the adjacent angles equal to two right angles HK is in the same straight line (I. 14), with HM. Again, because the straight line HG meets the parallels KM, FG, therefore the angle MHG is equal (I. 29) to the alternate angle HGF. To each of these equals, add the angle HGL. Therefore the two angles MHG, HGL are equal to the two angles HGF, HGL; but the two angles MHG, HGL are equal (I. 29) to two right angles. Therefore also the two angles HGF, HGL are equal (*Ax.* 1) to two right angles. Wherefore, as before, FG (I. 14) is in the same straight line with GL. Because KF is parallel to HG, and HG to ML, KF is parallel (I. 30) to ML. And KM has been proved parallel to FL. Therefore the figure KL is a parallelogram (*Def.* 36). But the parallelogram FH, is equal (*Const.*) to the triangle ABD, and the parallelogram GM to the triangle BDC. Therefore the whole parallelogram KL, is equal to the whole rectilineal figure ABCD. Wherefore the parallelogram KL has been described equal to the given rectilineal figure ABCD, and having the angle FKM equal to the given angle E. Q. E. F.

Corollary.—From this it is manifest how, to a given straight line, to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure; viz., by applying to the given straight line a parallelogram equal to the first triangle ABD (I. 44), and having an angle equal to the given angle, and constructing the rest of the figure as in this proposition.

By this proposition, also, upon a given straight line, a rectangle may be described, equal to a given rectilineal figure.

PROP. XLVI. PROBLEM.

To describe a square upon a given straight line.



Let AB be the given straight line. It is required to describe a square upon AB.

From the point A draw AC at right angles (I. 11) to AB. Make AD equal (I. 3) to AB. Through the point D draw DE parallel (I. 31) to AB, and through the point B draw BE parallel to AD. The figure AE is a square.

Because AE is a parallelogram (*Def.* 36) AB is equal (I. 34) to DE, and AD to BE. But BA is equal to AD. Therefore the four lines BA, AD, DE, EB are all equal to one another, and the parallelogram AE is equilateral. Again, because AD meets the parallels AB, DE, the two angles BAD, ADE, are equal (I. 29) to two right angles. But BAD is (*Const.*) a right angle. Therefore also ADE is a right angle. But the opposite angles of parallelograms (I. 34) are equal. Therefore each of the opposite angles ABE, BED, is a right angle, and the parallelogram AE is rectangular, that is, has all its angles right angles. And it has been proved to be equilateral. Therefore the figure AE is a square (*Def.* 30), and it is described upon the given straight line *AB*. *Q. E. F.*

Corollary 1.—Hence, every parallelogram that has one right angle, has all its right angles.

The construction of this problem may be effected, in accordance with Euclid's definition of a square, by Prop. XI. of this book. Euclid's corollary has also been anticipated.

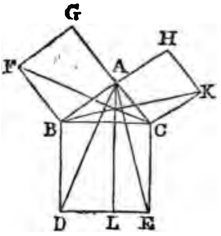
Corollary 2.—If two squares be equal, their sides, or the straight lines on which they are described, are equal.

Exercise.—If, in the sides of a square, points be taken at equal distances from its four angular points, in succession, the straight lines which join these points in the same order, will form a square. This square will be less than the original square in proportion to the distance of the four assumed from its angular points, until that distance be equal to half the side of the square, when the inscribed square (Def. 1, Book IV.) will be a *minimum*,—that is, the least possible square which can be thus inscribed.

PROP. XLVII. THEOREM.

In any right-angled triangle, the square described upon the side subtending the right angle, is equal to the squares described upon the two sides, containing the right angle.

Let A B C be a right-angled triangle, having the right angle B A C, The square described upon the side B C is equal to the squares described upon the two sides B A and A C.



Upon the side B C describe the square B E (I. 46), and on the sides B A and A C, the squares G B, and H C. Through A draw A L parallel to B D or C E (I. 31) and join A D and F C.

Because the angle B A C is a right angle (*Hyp.*), and that the angle B A G is a right angle (*Def.* 30), the two straight lines A C, A G, upon the opposite sides of A B, make with it, at the point A, the adjacent angles equal to two right angles. Therefore C A is (I. 14) in the same straight line with A G. For the same reason, B A and A H are in the same straight line. Because the angle D B C is equal to the angle F B A, each of them being a right angle. To each of these equals, add the angle A B C. Therefore the whole angle D B A is equal (*Ax.* 2) to the whole angle F B C. And because the two sides A B, B D, are equal to the two sides F B, B C, each to each, and the angle A B D to the angle F B C. Therefore the base A D is equal to the base F C (I. 4), and the triangle A B D to the triangle F B C. But the parallelogram B L is double of the triangle A B D (I. 41), because they are upon the same base B D, and between the same parallels B D, and A L. Also the square G B is double of the triangle F B C, because these are upon the same base F B, and between the same parallels F B and G C. Now, the doubles of equals are equal to one another (*Ax.* 6). Therefore the parallelogram B L is equal to the square G B. Similarly, by joining A E and B K, it can be proved that the parallelogram C L is equal to the square H C. Therefore the whole square B E is equal (*Ax.* 2) to the two squares G B and H C. Wherefore the square described upon the side B C opposite to the right angle B A C, is equal to the squares described upon the two sides A B and A C, containing Therefore, in any right-angled triangle, &c. Q. E. D.

In the construction of this proposition, Euclid has considered only one case or position of the three squares in relation to the sides of the triangle. But there are six different cases or positions of the three squares in relation to the sides of the triangle and to each other, as follows:—

1. The three squares on the *three exterior sides* of the triangle.
2. The three squares on the *three interior sides* of the triangle.
3. The two smaller squares on the *exterior sides*, and the greater on the *interior side*.
4. The two smaller squares on the *interior sides*, and the greater on the *exterior side*.
5. The greater and one of the smaller squares on the *exterior sides*, and the other on the *interior side*.
6. The greater and one of the smaller squares on the *interior side*, and the other on the *exterior side*.

It will be a useful and instructive exercise to the student to try and adapt the preceding demonstration to all these six different cases or positions of the squares.

Exercise 1.—In the figure of the preceding proposition, join G H, F D and K E, and prove that the three triangles, thus formed, are each equal to the triangle A B C, and to one another.

Exercise 2.—Prove that the three straight lines A L, F C, and K B, meet in the same point.

Exercise 3.—To describe a square equal to two given squares; and, consequently, to any given number of squares.

Exercise 4.—To describe a square equal to the difference between any two given squares; and, consequently, to the difference between a given square, and any number of given squares.

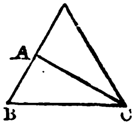
Exercise 5.—In any triangle, if a perpendicular be drawn from the vertex of any angle to the opposite side, the difference of the squares of the segments of that side is equal to the difference of the squares of the other two sides.

DEFINITION.—In any right-angled triangle, the side opposite to the right angle is called the *hypothenuse*, and the other sides, or legs, are called the *base* and the *perpendicular*; that is, if the one leg is the *base*, the other is the *perpendicular*, and conversely.

PROP. XLVIII. THEOREM.

If the square described upon one of the sides of a triangle, be equal to the squares described upon the other two sides of it; the angle contained by these two sides is a right angle.

Let the square described upon B C, one of the sides of the triangle A B C, be equal to the squares upon the other two sides, A B and A C. Then the angle B A C is a right angle.

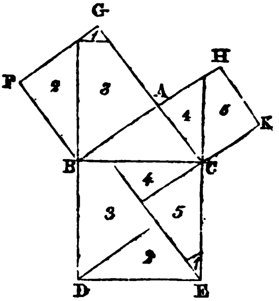


From the point A draw A D at right angles (I. 11) to A C. Make A D equal (I. 3) to A B, and join D C.

Because A D is equal to A B, the square of A D is equal to the square of A B. To each of these equals, add the square of A C. Therefore the squares of A D and A C are equal (*Ax.* 2) to the squares of A B and A C. But the squares of A D and A C are equal to the square of D C (I. 47), because the angle D A C is a right angle. And the square of B C (*Hyp.*) is equal to the squares of B A and A C. Therefore the square of D C is equal (*Ax.* 1) to the square of B C, and the side D C to the side B C. Again, because the side A D is equal to the side A B, and A C common to the two triangles D A C, B A C, the two sides D A, A C are equal to the two sides B A, A C, each to each, but the base D C has been proved to be equal to the base B C. Therefore the angle D A C is equal (I. 8) *to the angle* B A C. But D A C is a right angle. Therefore also B A C *is a right angle.* Therefore, if the square described upon, &c. Q. E. D.

From this proposition is deduced a very useful and practical mode of drawing a perpendicular to a given straight line from any point in the same. With a pair of compasses, mark off from the given point, five equal parts on the given straight line, marking them 1, 2, 3, 4, 5, at their extremities: 0, being marked at the given point. From the point marked 0, with radius equal to 4 of these parts describe an arc of a circle; and from the point marked 3, with radius equal to 5 of these parts, describe an arc of a circle intersecting the former arc. Join the point of intersection of the two arcs with the point marked 0, and the straight line thus drawn will be perpendicular to the given straight line at the given point, as required. For, joining the intersection of the arcs with the point marked 3, a triangle is formed; and the squares of its two sides 3 and 4 are together equal to the square of its side 5; because $9+16=25$. Therefore, the sides 3 and 4 contain a right angle, which is opposite to the side 5. The equimultiples of the numbers, 3, 4, and 5, will answer the same purpose equally well.

The following propositions may be added to this Book, chiefly as Exercises on the 47th proposition.



Prop. A. Theorem.—In any right angled triangle, the squares A F and AK, described on the two sides which contain the right angle B A C, may be so divided by straight lines, that the parts may be applied to the square B E described upon the side opposite the right angle, and made exactly to cover it.

In the adjacent figure, one method of proving this ingenious and important theorem by ocular demonstration is suggested; various other methods may be proposed, but that which requires the fewest number of parts, is most probably the best. The theoretical demonstration will form a highly useful exercise to the student.

Prop. B. Theorem.—In any triangle, if squares be described on the base, and on the other two sides; and if the perpendiculars to these sides, drawn from the extremities of the base, be produced to meet the opposite sides of the squares or those sides produced; the two rectangles cut off between these perpendiculars and the sides of the squares drawn from the extremities of the base, are together equal to the square of the base.

Prop. C. Theorem.—If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles opposite to either of the two equal sides be each a right angle, the triangles are equal to one another in all respects.

Prop. D. Theorem.—If two exterior angles of a triangle be bisected, and from the point of intersection of the bisecting lines, a straight line be drawn to the opposite angle, it will bisect that angle.

Prop. E. Theorem.—If from any point within or without any rectilineal figure, perpendiculars be drawn to each side, the sum of the squares of the alternate segments, reckoned from two adjacent angular points, are equal.

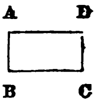
BOOK II.

DEFINITIONS.

I.

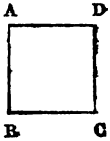
EVERY right-angled parallelogram is called a *rectangle*, and is said to be contained by any two of the straight lines which form one of its right angles.

Thus the rectangle A C is said to be contained by the two sides A B and B C which form the angle A B C; or by any of the other two sides which form the other angles, as B C and C D, C D and D A, or D A and A B; as it may have been described by means of any of these pairs of straight lines. For brevity's sake, the words *contained by* are generally omitted and a point is placed between the two sides to indicate that the rectangle is contained by these two sides.



A rectangle is said to be contained by two straight lines, when its adjacent sides are equal to those two straight lines, each to each.

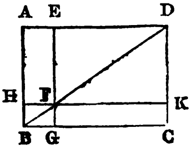
A square is said to be the square of a given straight line, when it is described upon that straight line; or, a straight line equal to it. Thus the square A C is said to be the square of A B, B C, C D, or D A; as it may have been described on either of these straight lines; it may also be said to be the square of any straight line equal to one of these straight lines.



II.

In every parallelogram, any of the parallelograms about a diagonal, together with the two complements, is called a gnomon.

"Thus the parallelogram H G together with the complements A F, F C, is a gnomon, which is more briefly expressed by the letters A G K, or E H C, at the opposite angles of the parallelograms which make up the gnomon." In like manner, the parallelogram E K together with the same two complements, A F, F C, is a gnomon; and is expressed by the letters A K G or C E H, at the opposite angles of he parallelograms which make up the gnomon.



AXIOMS.

I.

The whole is equal to all parts taken together.

This axiom, though not laid down by Euclid, is constantly employed in the demonstration of the propositions of this book.

II.

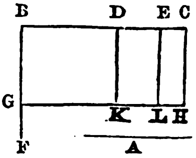
If two things be equal to one another, both taken together are the double of one of them.

This axiom, though not laid down, is assumed by Euclid in Prop. XLIII. of Book I. and various other places, but particularly in this book.

PROPOSITION I. THEOREM.

If there be two straight lines, one of which is divided into any number of parts; the rectangle contained by the two straight lines, is equal to the rectangle contained by the undivided line, and the several parts of the divided line.

Let A and BC be two straight lines; and let BC be divided into several parts at the points D and E. The rectangle contained by the two straight lines A and BC, is equal to the rectangles contained by the straight lines A and BD, A and DE, and A and EC.



From the point B, draw BF at right angles (I. 11) to BC. Make BG equal (I. 3) to A. Through G draw GH parallel (I. 31) to BC. And through D, E, and C, draw DK, EL, and CH parallel to BG. Then BH, BK, DL, and EH, are rectangles (I. *Def.* 36).

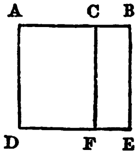
The rectangle BH is equal (II. *Ax.* 1) to the rectangles BK, DL, and EH. But the rectangle BH is contained by the straight lines A and BC, because GB is equal to A. The rectangle BK is contained by the straight lines A, and BD, because GB is equal to A. The rectangle DL is contained by the straight lines A and DE, because DK is equal to BG (I. 34) and BG is equal to A. In like manner, it is proved that the rectangle EH is contained by the straight lines A and EC. Therefore the rectangle contained by the straight lines A and BC, is equal to the several rectangles contained by the straight lines A and BD, A and DE, and A and EC. Wherefore, if there be two straight lines, &c. Q. E. D.

Corollary.—If two straight lines be divided each into any number of parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the several parts of the one and each of the several parts of the other.

PROP. II. THEOREM.

If a straight line be divided into any two parts, the rectangles contained by the whole and each of its parts, are together equal to the square of the whole line.

Let the straight line AB be divided into any two parts at the point C. The rectangles contained by AB, BC and AB, AC are together equal to the square of AB.



Upon AB describe (I. 46) the square AE. Through C draw CF parallel (I. 31) to AD or BE.

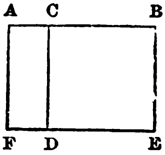
The square AE is equal (II. *Ax.* 1) to the rectangles AF and CE. But AE is the square of AB. The rectangle AF is contained by BA, AC; because DA is equal to AB. The rectangle CE is contained by AB, BC, because BE is equal to AB. Therefore the rectangles contained by AB, AC and AB, BC are equal (*Ax.* 1) to the square of AB. Wherefore if a straight line, &c. Q. E. D.

This proposition is merely a corollary to the first proposition; for, when the two *straight lines* mentioned in its enunciation are equal, their rectangle is a square.

PROP. III. THEOREM.

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the rectangle contained by the two parts, together with the square of the foresaid part.

Let the straight line A B be divided into any two parts at the point C. The rectangle AB.BC is equal to the rectangle A C.CB, together with the square of B C.



Upon B C describe the square CE (I. 46). Produce ED to F. Through A draw A F parallel (I. 31) to CD or BE.

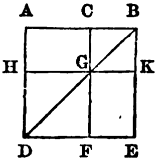
The rectangle AE is equal (II. *Ax.* 1) to the rectangles AD and CE. But AE is the rectangle contained by AB, BC, because BE is equal (*Const.*) to BC. The rectangle AD is contained by AC, CB, because CD is equal to CB. And DB is the square of B C. Therefore the rectangle A B.B C is equal to the rectangle A C.C B, together with the square of B C. If therefore a straight line be divided, &c. Q. E. D.

This proposition is only another corollary to the first proposition; for, when the undivided straight line mentioned in its enunciation, is equal to one of the parts of the divided straight line, their rectangle becomes a square.

PROP. IV. THEOREM.

If a straight line be divided into any two parts, the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the parts.

Let the straight line AB be divided into any two parts at C. The square of A B is equal to the squares of A C and C B, together with twice the rectangle A C.CB.



Upon A B describe the square AE (I. 46). Join BD. Through C draw CGF parallel to AD or BE (I. 31), and through G draw HGK parallel to AB or DE.

Because CF is parallel to AD, and BD falls upon them, the exterior angle BGC is equal (I. 29) to the interior and opposite angle ADB. But the angle A D B is equal to the angle A B D (I. 5), because B A is equal to AD, being sides of a square. Therefore the angle CGB is equal (I. *Ax.* 1) to the angle CBG, and the side BC (I. 6) to the side CG. But CB is equal (I. 34) also to GK, and CG to BK. Therefore the figure CK is equilateral. Again, because CG is parallel to BK, and CB meets them, the angles KB C, GCB are equal (I. 29) to two right angles. But the angle KBC is a right angle (*Const.*) Therefore GCB is a right angle. And the angles CGK, GKB, opposite to these, are also (I. 34) right angles. Therefore CK is rectangular. But it is also equilateral. Therefore it is a square, and it is described upon the side CB. For the same reason H F is a square, and it is described upon the side HG, which is (I. 34) equal to AC. Therefore the figures *HF* and *CK* are the squares of AC and CB. Because the complement

AG is equal (I. 43) to the complement GE; and AG is the rectangle contained by AC.CB, for GC is equal to CB. Therefore GE is also equal to the rectangle AC.CB. Wherefore AG and GE are equal to twice the rectangle AC.CB. Also HF and CK are the squares of AC and CB. Therefore the four figures HF, CK, AG and GE, are equal to the squares of AC and CB, with twice the rectangle AC.CB. But the figures HF, CK, AG and GE make up the whole figure AE, which is the square of AB. Therefore the square of AB is equal to the squares of AC and CB, and twice the rectangle AC.CB. Wherefore, if a straight line be divided, &c. Q. E. D.

COR. 1.—From the demonstration, it is manifest, that the parallelograms about the diameter of a square, are likewise squares.

Corollary 2.—The square of any straight line is equal to four times the square of half the straight line.

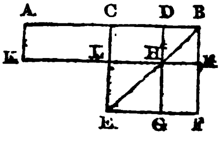
The demonstration of the preceding proposition might have been shortened by the application of some corollaries to the propositions in Book I. It will be a useful exercise for the student to discover this abridgment himself.

The following demonstration is founded on the preceding propositions of this book. Because AB is divided into any two parts at C, the square of AB is equal (II. 2) to the two rectangles AB.AC, and AB.BC. But because AB is divided into any two parts at C, the rectangle AB. AC is equal (II. 3) to the rectangle AC.CB, and the square of AC; also the rectangle AB.BC is equal to the rectangle AC.CB, and the square of CB. Adding these equals, the rectangles AB.AC and AB.BC, are together equal to twice the rectangle AC.CB, and the squares of AC and CB. But it has been proved that the square of AB is equal to the same two rectangles. Therefore the square of AB is equal (I. *Ax.* 1) to the squares of AC and CB, and twice the rectangle AC.CB.

PROP. V. THEOREM.

If a straight line be divided into two equal parts, and also into two unequal parts; the rectangle contained by the unequal parts, together with the square of the line between the points of section (or division), is equal to the square of half the line.

Let the straight line AB be divided into two equal parts at the point C, and into two unequal parts at the point D. The rectangle AD,DB, together with the square of CD, is equal to the square of CB.



Upon CB describe (I. 46) the square CF. Join BE. Through D draw DHG parallel to CE or BF (I. 31); and through H draw KLM parallel to CB or EF. Also through A draw AK parallel to CL or BM.

Because the complement CH is equal (I. 43) to the complement HF. To each of them, add DM. Therefore the whole CM is equal to the whole DF. But CM is equal to AL (I. 36) because AC is equal to CB. Therefore AL is equal to DF. To each of these equals, add CH. Therefore the whole AH is equal to DF and CH; that is, to the gnomon CMG. But AH is the rectangle AD.DB, for DH is equal to DB. Therefore the gnomon CMG is equal to the rectangle *AD.DB. To each of* these equals add LG, which is equal (II. 4. *Cor.*) *to the square of CD.* Therefore the gnomon CMG, together with LG, is equal to the rectangle AD.BD, together with the square of CD. But the gnomon CMG and LG make up the whole figure CF, which is the square of CB. Therefore the rectangle AD.DB, together with the square of CD, is equal to the square of CB. Wherefore, if a straight line, &c. Q. E. D.

Another Demonstration: Because CB is divided into any two parts at D, the rectangle CD.DB, and the square of DB are together equal (II. 3) to the rectangle CB.BD, or to the rectangle AC.DB; for, AC is equal (*Const.*) to CB. To each of these equals, add the rectangle CD.DB. Therefore twice the rectangle CD.DB, and the square of DB are together equal to the rectangles AC.DB, and CD.DB, which are equal (II. 1) to the rectangle AD.DB. Again, to each of these equals add the square of CD. Therefore the squares of CD and DB, and twice the rectangle CD.DB are together equal to the rectangle AD.DB, and the square of CD. But the squares of CD and DB and twice the rectangle CD.DB, are together equal (II. 3) to the square of CB. Therefore the rectangle AD.DB, and the square of CD, are together equal to the square of CB.

Corollary 1.—From this proposition it is manifest, that the difference of the squares of two unequal lines AC and CD, is equal to the rectangle contained by their sum AD, and their difference DB.

Corollary 2.—The rectangle contained by the segments of any straight line, is a maximum when the point of section is the middle point.

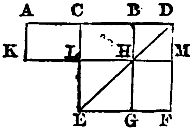
Corollary 3.—The sum of the squares of the two parts into which a straight line is divided is a minimum, when it is bisected.

Corollary 4.—The square of one of the legs of a right-angled triangle is equal to the rectangle contained by the sum and the difference of the hypotenuse and the other leg.

PROP. VI. THEOREM

If a straight line be bisected, and produced to any point, the rectangle contained by the whole line thus produced, and the part of it produced, together with the square of half the line bisected, is equal to the square of the straight line which is made up of the half and the part produced.

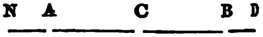
Let the straight line AB be bisected at C, and produced to the point D. The rectangle AD.DB, together with the square of CB, is equal to the square of CD.



Upon CD describe the square CF (I. 46). Join DE. Through B draw BHG parallel to CE or DF (I. 31); and through H draw KLM parallel to AD or EF. Also, through A draw AK parallel to CL or DM.

Because AC is equal to CB, the rectangle AL is equal (I. 36) to the rectangle CH. But CH is equal (I. 43) to HF. Therefore AL is equal to HF. To each of these equals add CM. Therefore the whole AM is equal to the gnomon CMG. But AM is the rectangle AD.DB, because DM is equal (II. 4. *Cor.*) to DB. Therefore the gnomon CMG is equal to the rectangle AD.DB. To each of these equals add LG, which is equal to the square of CB. Therefore the rectangle AD.DB, together with the square of CB, is equal to the gnomon CMG, and LG. But the gnomon CMG and LG make up the square CF, which is the square of CD. Therefore the rectangle AD.DB, together with the *square of CB*, is equal to the square of CD. Wherefore a straight *line, &c.* Q. E. D.

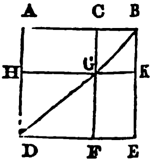
Another Demonstration: Produce C A to N, making CN equal to C D. To these equals, add the equals CA and CB respectively. Therefore NB, is equal to A D. But the rectangle N B B D and the square of C B is equal (II. 5) to the square of CD. And it has been proved that AD is equal to NB. Therefore the rectangle A D.D B, and the square of C B is equal to the square of CD.



PROP. VII. THEOREM.

If a straight line be divided into any two parts, the squares of the whole line, and of one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.

Let the straight line AB be divided into any two parts at the point C. The squares of AB and BC are equal to twice the rectangle AB.BC, together with the square of AC.



Upon AB describe the square AE (I. 46) and join BD. Through C, draw CF parallel to AD or BE, cutting BD in G (I. 31), and through G draw HGK parallel to AB or DE.

Because AG is equal to GE (I. 43); to each of them add CK. Therefore the whole AK is equal to the whole CE; and AK and CE together, are double of AK. But AK and CE are the gnomon AKF and the square CK. Therefore the gnomon AKF and the square CK are together double of AK. But twice the rectangle AB.BC, is double of AK, for BK is equal (II. 4 *Cor.*) to BC. Therefore the gnomon AKF and the square CK, are together equal to twice the rectangle AB.BC. To each of these equals, add HF, which is equal to the square of AC. Therefore the gnomon AKF, and the squares CK and HF, are equal to twice the rectangle AB.BC, and and the square of AC. But the gnomon AKF, and the squares CK and HF, make up the squares AE and CK, which are the squares of AB and BC. Therefore the squares of AB and BC are equal to twice the rectangle AB.BC, together with the square of AC. Wherefore, if a straight line, &c. Q. E. D.

Otherwise.—Because A B is divided into any two parts at C, the square of A B is equal (II. 4) to the squares of A C and C B, and twice the rectangle A C.C B. To these equals, add the square of C B. Therefore the squares of A B and C B are together equal to the square of A C, twice the square of C B, and twice the rectangle A C.C B. But the rectangle A B. B C is equal (II. 3) to the square of C B, and the rectangle A C.C B. Therefore twice the rectangle A B.B C is equal (I. *Ax.* 6) to twice the square of C B, and twice the rectangle A C.C B. To these equals, add the square of A C. Therefore the square of A C and twice the rectangle A B.B C are together equal (I. *Ax.* 2) to the square of A C, twice the square of CB and twice the rectangle A C.CB. But it was proved that the squares of A B and C B are equal to the square of A C, twice the square of C B, and twice the rectangle A C.C B. Therefore the squares of A B and C B are together equal (I. *Ax.* 1) to twice the rectangle A B.B C and the square of A C.

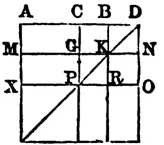
Corollary 1.—The square of the difference of two straight lines is equal to the difference between the sum of their squares and twice the rectangle contained by them.

Corollary 2.—The square of the same of two straight lines is equal to the square *of their difference*, together with four times the rectangle contained by them.

PROP. VIII. THEOREM.

If a straight line be divided into any two parts, four times the rectangle contained by the whole line, and one of the parts, together with the square of the other part, is equal to the square of the straight line, which is made up of the whole and that part.

Let the straight line AB be divided into any two parts at the point C. Four times the rectangle AB.BC, together with the square of AC, are equal to the square of the straight line made up of AB and BC together.



Produce AB to D, making (I. 3) BD equal to BC. Upon AD describe (I. 46) the square AF. Join its opposite points DE. Through the points B and C, draw the straight lines (I. 31) BKL and CPH parallel to AE or DF. Through the points K and P, where they meet the diagonal, draw MGN and XPO parallel to AD or EF.

Because CB is equal to BD (*Const.*) and also to GK (I. 34), the squares (II. 4 *Cor.*) GR and BN are equal. Because the sides of these squares are all equal, and all their adjacent angles are right angles (I. 13). Therefore the complements CK and KO are (I. 43) equal squares, and the four squares (I. 36) CK, BN, GR and KO are all equal. Because CG, and GP, are sides of equal squares, the rectangles AG and MP are (I. 36) equal. For the same reason, the rectangles PL and RF are equal. But the rectangle MP is (I. 43) equal to the rectangle PL. Therefore the four rectangles AG, MP, PL, and RF are all equal. Because the four squares CK, BN, GR, and KO are four times CK; and the four rectangles AG, MP, PL, and RF are four times AG. Therefore the gnomon AOH is four times the rectangle AK; that is, four times the rectangle AB. BC, because BK is equal to BC. To these equals, add the square of AC, or its equal the square XH (I. 34). Therefore four times the rectangle AB. BC and the square of AC, are together equal to the gnomon AOH and the square XH; that is, to the square AF. But the square AF is the square of AD, or of AB and BC together. Therefore four times the rectangle AB. BC and the square of AC, are together equal to the square of the straight line made up of AB and BC together. Q. E. D.

Dr. Thomson, in his edition, objects to this demonstration on the ground that it tacitly assumes the truth of the first proposition of Book V. This objection may be removed by the following demonstration:—Because the triangle ADE is equal (I. 34) to the triangle FDE, and the triangle XPE equal to the triangle HPE. Therefore the remaining figure ADPX is equal (I. *Ax.* 3) to the remaining figure FDPH; and the gnomon AOH is double (II. *Ax.* 2) the figure ADPX. Because CB is equal (*Const.*) to BD, the square GR is equal (I. 34) to the square BN, and the triangle BDK to the triangle KPR (I. *Ax.* 7). To these equals, add the figure ABKPX. Therefore the figure ADPX is equal to the rectangle AR. But the rectangle AR is double the rectangle AK, because BK is equal to KR. Therefore the figure ADPX is double the rectangle AK. But it has been proved that the gnomon AOH is double the figure ADPX. Therefore the gnomon AOH is four *times the rectangle* AK; that is, four times the rectangle AB. BC; because BK *is equal to* BC. The rest of this demonstration is the same as that of the preceding; *viz.*, "To these equals add. &c"

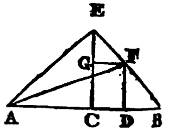
Otherwise.—Produce A B to D making B D equal to B C. Because the square of C D is equal (II. 4 *Cor.* 2) to four times the square of C B; and twice the rectangle A C. C D is equal (II. *Ax.* 2) to four times the rectangle A C. C B. Therefore, adding these equals, the square of C D and twice the rectangle A C. C D are together equal to four times the square of C B, and four times the rectangle A C. C B. But the rectangle A B. B C is equal (II. 3) to the square of C B and the rectangle A C. C B. Therefore four times the rectangle A B. B C is equal to four times the square of C B and four times the rectangle A C. C B. But it has been proved that the square of C D, and twice the rectangle A C. C D are together equal to the same magnitudes. Therefore the square of C D and twice the rectangle A C. C D are together equal to four times the rectangle A B. B C. To these equals add the square of A C. Therefore the squares of A C and C D, and twice the rectangle A C. C D, are together equal to four times the rectangle A B. B C and the square of A C. But the squares of A C and C D, and twice the rectangle A C. C D are equal (II. 4) to the square of A D. Therefore four times the rectangle A B. B C and the square of A C, are together (I. *Ax.* 1) equal to the square of A D, that is, of the straight line made up of A B and B C together.

PROP. IX. THEOREM.

If a straight line be divided into two equal, and also into two unequal parts; the squares of the two unequal parts are together double of the square of half the line, and of the square of the line between the points of section.

Let the straight line A B be divided into two equal parts at the point C and into two unequal parts at the point D. The squares of A D and and D B together, are double the squares of A C and C D together.

From the point C draw (I. 11) C E at right angles to A B. Make C E equal (I. 3) to A C, and join E A, E B. Through D draw D F parallel (I. 31) to C E, meeting E B in F. Through F draw F G parallel to B A, and join A F.



Because A C is equal to C E, the angle E A C is equal (I. 5) to the angle A E C. Because A C E is a right angle, the two other angles A E C, E A C of the triangle A E C are together equal (I. 32) to a right angle. But they are equal to one another. Therefore each of them is half a right angle. For the same reason, each of the angles C E B, E B C is half a right angle. Therefore the whole angle A E B is a right angle. Because the angle G E F is half a right angle, and E G F is a right angle, being equal (I. 29) to the interior and opposite angle E C B. Therefore the remaining angle E F G is half a right angle. Wherefore the angle G E F is equal to the angle E F G, and the side E G (I. 6) to the side G F. Again, because the angle at B is half a right angle, and F D B is a right angle, being equal (I. 29) to the interior and opposite angle E C B. Therefore the remaining angle B F D is half a right angle. Wherefore the angle at B is equal to the angle B F D, and the side D F (I. 6) to the side D B. Because A C is equal to C E, the square of A C is equal to the square of C E. Therefore the squares of A C and C E are double of the square of A C. But the square of A E is equal (I. 47) to the squares of A C and C E. Therefore the square of A E is double of the square of A C. Again, because E G is equal to G F, the square of E G is equal to the square of G F. Therefore the squares of E G and G F are double of the square of G F. But the square of E F is equal (I. 47) to the squares of E G and G F. Therefore the square of

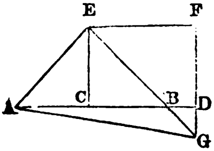
EF is double of the square of GF. But GF is equal (I. 34) to CD. Therefore the square of EF is double of the square of CD. But it has been proved that the square of AE is double of the square of AC. Therefore the squares of AE and EF are double of the squares of AC and CD. But the square of AF is equal (I. 47) to the squares of AE and EF. Therefore the square of AF is double of the squares of AC and CD. But the squares of AD and DF are equal (I. 47) to the square of AF. Therefore the squares of AD and DF are double of the squares of AC and CD. But DF is equal to DB. Therefore the squares of AD and DB are double of the squares of AC and CD. If, therefore, a straight line be divided, &c. Q. E. D.

Otherwise.—Because BC is divided into any two parts at D, the square of BD, and twice the rectangle BC. CD are together equal (II. 7) to the squares of BC and CD. But AC is equal (*Const.*) to BC. Therefore, the square of BD and twice the rectangle AC. CD are together equal to the squares of AC and CD. But the square of AD is equal (II. 4) to the squares of AC and CD, and twice the rectangle AC. CD. Therefore adding these equals, the squares of AD and BD, and twice the rectangle AC. CD are together equal (I. *Ax.* 2) to twice the squares of AC and CD, and twice the rectangle AC. CD. From these equals, take away twice the rectangle AC. CD. Therefore the squares of AD and BD are equal (I. *Ax.* 3) to twice the squares of AC and CD.

PROP. X. THEOREM.

If a straight line be bisected, and produced to any point, the squares of the whole line thus produced, and of the part of it produced, are together double of the square of half the line bisected, and of the square of the line made up of the half and the part produced.

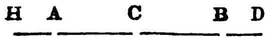
Let the straight line AB be bisected in C, and produced to the point D. The squares of AD and DB, are together double of the squares of AC and CD.



From the point C draw CE (I. 11) at right angles to AB. Make CE equal (I. 3) to AC or CB, and join AE and EB. Through E draw EF parallel to AB (I. 31), and through D draw DF parallel to CE. Because the straight line EF meets the parallels CE, FD, the two angles CEF, EFD are equal (I. 29) to two right angles. Therefore the two angles BEF, EFD are less than two right angles. But straight lines, which with another straight line make the interior angles upon the same side less than two right angles, will meet (*Ax.* 12) if produced far enough. Therefore EB and FD will meet, if produced towards B and D. Let them meet in G, and join AG.

Because AC is equal to CE, the angle CEA is equal (I. 5) to the angle EAC. Because ACE is a right angle, each of the angles CEA and EAC is half a right angle (I. 32). For the same reason, each of the angles CEB and EBC is half a right angle. Therefore the whole angle AEB is a right angle. Because EBC is half a right angle, the vertical angle DBG is also (I. 15) half a right angle. But BDG is a right angle, being equal (I. 29) to the alternate angle DCE. Therefore the remaining angle DGB is half a right angle. Wherefore the *angle DGB* is equal to the angle DBG, and the side BD (I. 6) to th side DG. Again, because EGF is half a right angle, and the angle at F is a right angle, being equal (I. 34) to the opposite angle ECD, the remaining angle FEG is half a right angle. Therefore the angle FEG is equal to the angle EGF, and the side GF (I. 6) to the side FE. Because EC is equal to CA, the square of EC is equal to the square of CA. Therefore the squares of EC and CA are double of the square of CA. But the square of EA is equal (I. 47) to the squares of EC and CA. Therefore the square of EA is double of the square of AC. Again, because GF is equal to EF, the square of GF is equal to the square of EF. Therefore the squares of GF and FE are double of the square of EF. But the square of EG is equal (I. 47) to the squares of GF and EF. Therefore the square of EG is double of the square of EF. But EF is equal (I. 34) to CD. Wherefore the square of EG is double of the square of CD. But it was proved that the square of EA is double of the square of AC. Therefore the squares of EA and EG are double of the squares of AC and CD. But the square of AG is equal (I. 47) to the squares of EA and EG. Therefore the square of AG is double of the squares of AC and CD. But the squares of AD and DG are equal to the square of AG. Therefore the squares of AD and DG are double of the squares of AC and CD. But DG is equal to DB. Therefore the squares of AD and DB are double of the squares of AC and CD. Wherefore, if a straight line, &c. Q. E. D.

Otherwise.—Produce CA, making CH equal (I. 3) to CD. To these equals, add the equals CB and CA, respectively. Therefore HB is equal (I. *Ax.* 2) to AD. Because the squares of HB and BD are together (II. 9) double of the squares of AC and CD. But HB is equal to AD. Therefore the squares of AD and BD are together double of the squares of AC and CD together.

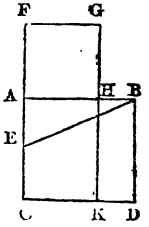


PROP. XI. PROBLEM.

To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts, shall be equal to the square of the other part.

Let AB be the given straight line. It is required to divide AB into two parts, so that the rectangle contained by the whole and one of the parts, shall be equal to the square of the other part.

Upon AB describe (I. 46) the square CB. Bisect AC in E (I. 10), and join BE. Produce CA to F, and make EF equal (I. 3) to EB. Upon AF describe (I. 46) the square FH. The straight line AB is divided in H, so that the rectangle AB. BH is equal to the square of AH. Produce GH to meet CD in K.



Because, the straight line AC is bisected in E, and produced to F, the rectangle CF. FA together with the square of AE, is equal (II. 6) to the square of EF. But EF is equal to EB. Therefore the rectangle CF. FA, together with the square of AE, is equal to the square EB. But the squares of BA and AE are equal (I. 47) to the square of EB. Therefore the rectangle CF. FA, together with the square of AE, to the squares of BA and AE. From these equals, take away the square of A E, which is common to both. Therefore the rectangle C F. F A is equal to the square of B A. But the rectangle F K is the rectangle contained by C F. F A, because F A is equal to F G. And A D is the square of A B. Therefore the rectangle F K is equal to the square A D. From these equals, take away the common part A K. Therefore the remainder F H is equal to the remainder H D. But H D is the rectangle contained by A B.B H, because A B is equal to B D. And F H is the square of A H. Therefore the rectangle A B. B H, is equal to the square of A H. Wherefore the straight line A B is divided in H, so that the rectangle A B.B H is equal to the square of A H. Q. E. F.

Dr. Thomson in his edition, has made an improvement in the construction of this problem, by cutting off at once from A B, a part A H equal to A F, and then completing the square F H for the demonstration. This removes the hiatus in the demonstration of Euclid, which does not show that A H and A B, the sides of the squares F H and A D, must coincide. It creates another, however, for it is not shown by Dr. Thomson that A B is the greater of the two straight lines A B and A F. But the same defect lurks in the demonstration of Euclid.

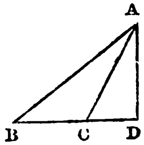
Corollary 1.—In the figure to this proposition, the straight line C F is cut at the point A, similarly to the straight line A B at the point H.

Corollary 2.—If a straight line be divided, so that the rectangle contained by the whole and the smaller segment is equal to the square of the greater, the greater segment will be similarly divided by cutting off from it a part equal to the smaller segment; and the smaller segment will be similarly divided by cutting off from it a part equal to their difference, and so on continually.

PROP. XII. THEOREM.

In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle, is greater than the squares of the sides containing the obtuse angle, by twice the rectangle contained by the side to which, when produced, the perpendicular is drawn, and the straight line intercepted between the perpendicular and the obtuse angle.

Let A B C be an obtuse-angled triangle, having the obtuse angle A C B; and from the point A, let A D be drawn perpendicular to B C produced. The square of A B is greater than the squares of A C and C B, by twice the rectangle B C. C D.

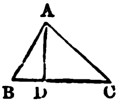


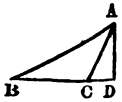
Because the straight line B D is divided into two parts at the point C, the square of B D is equal (II. 4) to the squares of B C and C D, and twice the rectangle B C. C D. To each of these equals, add the square of D A. Therefore the squares of B D and D A are equal to the squares of B C, C D and D A, and twice the rectangle B C.C D. But the square of B A is equal (I. 47) to the squares of B D and D A. And the square of C A is equal to the squares of C D and D A. Therefore the square of B A is equal to the squares of B C and C A, and twice the rectangle B C. C D; that is, the square of B A is greater then the squares of B C and C A, by twice the rectangle B C.C D. Therefore, in obtuse-angled triangles, &c. Q. E. D.

PROP. XIII. THEOREM.

In every triangle, the square of the side subtending either of the acute angles, is less than the squares of the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the acute angle and the perpendicular drawn to it from the opposite angle.

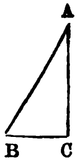
Let ABC be any triangle, and the angle at B one of its acute angles; and to B C, one of the sides containing it, draw the perpendicular AD from the opposite angle at A (I. 12). The square of AC opposite to the angle B, is less than the squares of CB and BA, by twice the rectangle CB.DB.





Because the straight line CB or BD, is divided into two parts at D or at C, the squares of CB and BD are equal (II. 7) to twice the rectangle CB.BD, and the square of DC. To each of these equals, add the square of AD. Therefore the squares of CB, BD, and DA, are equal to twice the rectangle CB.BD, and the squares of AD and DC. But the square of AB is equal (I. 47) to the squares of BD and DA. And the square of AC is equal to the squares of AD and DC. Therefore the squares of CB and BA are equal to the square of AC, and twice the rectangle CB. BD; that is, the square of AC is less than the squares of CB and BA, by twice the rectangle CB. BD.

When the perpendicular coincides with AC, the leg BC is the straight line between the perpendicular and the acute angle at B; and it is manifest, that the squares of AB and BC, are equal (I. 47) to the square of AC, and twice the square of BC. Therefore in any triangle, &c. Q. E D.

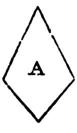


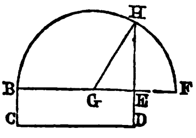
Corollary.—The square of any side of a triangle is greater than equal to, or less than the squares of the other two sides according as the angle opposite to that side is greater than, equal to or less than a right angle; and the difference, where it exists, is twice the rectangle contained by either of the other sides, and the straight line intercepted between the vertex of that angle and a perpendicular drawn to the remaining side from its opposite angle.

PROP. XIV. PROBLEM.

To describe a square that shall be equal to a given rectilineal figure.

Let A be the given rectilineal figure. It is required to describe a square that shall be equal to A.





Describe the rectangular parallelogram BD equal (I. 45) to the rectilineal figure A. If the sides BE and ED of the rectangle BD are equal to one another, it is a square, and what was required is done. But if they are not equal, produce one of them BE to F, and make (I. 3) EF equal to ED. Bisect (I. 10) BF in G. From the centre G, at the distance GB or GF, describe the semicircle BHF. Produce DE to meet the circumference in H. The square described upon EH is equal to the given rectilineal figure A. Join GH.

Because the straight line BF is divided into two equal parts at G, and into two unequal parts at E; the rectangle BE. EF, together with the square of EG, is equal (II. 5) to the square of GF. But GF is equal (*Def.* 15) to GH. Therefore the rectangle BE. EF, together with the square of EG, is equal to the square of GH. But the squares of HE and EG are equal (I. 47) to the square of GH. Therefore the rectangle BE. EF, together with the square of EG, is equal to the squares of HE and EG. From these equals, take away the square of EG, which is common to both. Therefore the rectangle BE. EF is equal to the square of HE. But the rectangle contained by BE. EF is the parallelogram BD, because EF is equal to ED. Therefore BD is equal to the square of EH. But BD is equal (*Const.*) to the rectilineal figure A. Therefore the square of EH is equal to the rectilineal figure A. Wherefore the square described upon EH, is equal to the given rectilineal figure A. Q. E. F.

Corollary.—The square of a perpendicular drawn from any point in a circle to its diameter is equal to the rectangle contained by the segments into which it divides the diameter.

The following Propositions and Corollaries are added to this Book in some editions of Euclid. As they are comparatively easy to the student who has mastered the first and second books, we give them as exercises.

Prop. A. Theorem.—The squares of any two sides of a triangle are together double of the squares of half the third side and of the straight line drawn from the opposite angle bisecting that side.

Prop. B. Theorem.—The squares of the two diagonals of a parallelogram are together equal to the squares of its four sides.

Prop. C. Theorem.—The squares of the four sides of a trapezium are together equal to the squares of its two diagonals, and four times the square of the straight line which joins the points of the bisection of the diagonals.

Prop. D. Problem.—To divide a given straight line into two parts so that the rectangle contained by its segments shall be equal to a given square, not greater than the square of half the given straight line.

Prop. E. Problem.—To produce a given straight line, so that the rectangle contained by the whole line thus produced and the part produced, shall be equal to a given square.

BOOK III.

DEFINITIONS.

I.

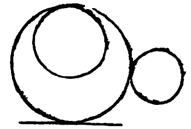
EQUAL circles are those of which the diameters are equal, or those from the centres of which the [radii, or] straight lines drawn to the circumferences are equal.

"This is not a definition, but a theorem, the truth of which is evident; for, if the circles be applied one to another, so that their centres coincide, the circles must likewise coincide, since the straight lines drawn from the centres are equal."

Concentric circles are those which have the same centre.

II.

A straight line is said to touch a circle, when it meets the circumference, and being produced does not cut the circle, that is, does not intersect the circumference.



If a straight line *touch* a circle in the sense thus defined, it is called a *tangent* (touching) to the circle; but one which *cuts* the circle, is called a *secant* (cutting) to the circle.

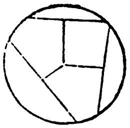
III.

Circles are said to touch one another when their circumferences meet in any point, but do not cut one another.

The point in the circumference of a circle, where a straight line or another circle touches it, is called *the point of contact* (touch).

IV.

Straight lines are said to be equally distant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal.



The straight lines spoken of in this definition are usually called *chords* (strings) because they stretch from one point of the circumference to another. If the perpendiculars drawn to them from the centre, be produced to meet the circumference, the parts of these perpendiculars between the chords and the circumference are called *sagittæ* (arrows).

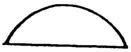
V.

And the straight line which has the greater perpendicular drawn to it, is said to be farther from the centre.

The straight line, or chord, which has the less perpendicular drawn to it, is said *to be nearer to the* centre.

VI.

A segment of a circle is the figure contained by a straight line or chord, and the arc, or part of the circumference which it cuts off.



Every chord, except a diameter, divides a circle into two unequal segments, the one greater and the other less than a semicircle. For brevity's sake, let these segments be called *supplementary* to each other.

VII.

[The angle of a segment is that which is contained by the straight line and the circumference.]

This definition appears to be an interpolation. It is not used and need not be remembered.

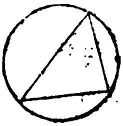
VIII.

An angle in a segment is the angle contained by two straight lines drawn from any point in the circumference of the segment, to the extremities of the straight line which is the base of the segment.

The meaning of this definition is, that if from the extremities of the chord of a segment, two other chords be drawn to any point in the arc of the segment, the angle formed between these two chords is called *the angle in the segment.*

IX.

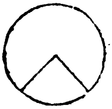
An angle is said to insist or stand upon the circumference intercepted between the straight lines that contain the angle.



The meaning of this definition is that *the angle in a segment* is said to stand upon the arc of its supplementary segment.

X.

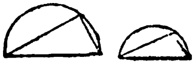
A sector of a circle is the figure contained by two (radii, or) straight lines drawn from the centre, and the arc, or part of the circumference between them.



The two radii, except when they are in the same straight line, and thus form a diameter, divide a circle into two unequal sectors, the one greater and the other less than a semicircle. These may also be called *supplementary* sectors. Sectors receive names, sometimes indicative of the part which they form of the entire circle, as a *quadrant*, or the fourth part of a circle; a *sextant*, or sixth part; and an *octant*, or eighth part.

XI.

Similar segments of circles are those in which the angles are equal, or which contain equal angles.



AXIOMS.

I.

If a point be taken between the centre of a circle and its circumference, that point is within the circle; and if a point be taken beyond ***the circumference***, it is without the circle.

This axiom is tacitly assumed by Euclid in this Book.

II.

If two magnitudes be doubles of two other magnitudes, each of each, the sum of the first two is double the sum of the other two.

III.

If two magnitudes be doubles of two other magnitudes, each of each, the difference between the first two is double the difference between the other two.

These two axioms are not given by Euclid; they are necessary, however, to complete the logical demonstration of the 20th proposition of this Book. Dr. Thomson, in his edition gives Playfair's demonstration of them.

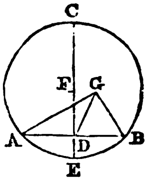
PROP. I. PROBLEM.

To find the centre of a given circle.

Let A B C be the given circle. It is required to find its centre.

Take any two points A and B, in the circumference, and join A B.

Bisect the straight line AB (I. 10) at D. From the point D, draw D C at right angles (I. 11) to A B. Let C D meet the circumference in C and E; and bisect C E in F. The point F is the centre of the circle A B C. For if it be not, let, if possible, G, a point not in C E, be the centre, and join G A, G D and G B.



Because D A is equal (*Const.*) to D B, and D G common to the two triangles A D G, B D G, the two sides A D, D G, are equal to the two sides B D, D G, each to each. But the base G A is equal (I. *Def.* 15) to the base G B, because they are drawn from the centre G. Therefore the angle A D G is equal (I. 8) to the angle G D B. But when a straight line standing upon another straight line makes the adjacent angles equal to one another, each of these angles is a right angle (I. *Def.* 10). Therefore the angle G D B is a right angle. But F D B is likewise (*Const.*) a right angle. Wherefore the angle F D B is equal (I. *Ax.* 1) to the angle G D B; that is, the greater equal to the less, which is impossible. Therefore G is not the centre of the circle A B C. In the same manner it can be shown that no other point out of C E is the centre. Because C E is bisected in F, any other point in C E divides it into unequal parts, and cannot be the centre. Therefore no other point but F can be the centre. Wherefore F is the centre of the circle A B C. Q. E. F.

COR. From this it is manifest, that if in a circle, a straight line bisects another at right angles, the centre of the circle is in the line which bisects the other.

The simplest mode of finding the centre of a circle, is to draw any two chords and bisect them at right angles by two straight lines. The intersection of these straight lines will be the centre, by the preceding corollary.

***Exercise* 1.—Given the arc of a circle, to find the centre of the arc, that is, of the circle of which it is an arc.**

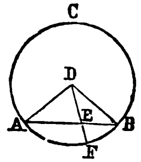
Exercise* 2.—To describe a circle that shall pass through any three points which are *not in the same straight line.

PROP. II. THEOREM.

If any two points be taken in the circumference of a circle, the straight line which joins them lies within the circle.

Let A B C be a circle, and A and B, any two points in the circumference. The straight line drawn from A to B lies within the circle.

Find D the centre of the circle A B C (III. 1), and join D A and D B. In the arc A B, take any point F; join D F, and let it meet the straight line A B in E.



Because D A is equal (I. *Def.* 15) to D B, the angle D A B is equal (I. 5) to the angle D B A. Because A E, a side of the triangle D A E, is produced to B, the exterior angle D E B is greater (I. 16) than the interior and opposite angle D A E. But the angle D A E was proved to be equal to the angle DBE. Therefore the angle DEB is also greater than the angle DBE. But the greater side is opposite (I. 19) to the greater angle. Therefore D B is greater than D E. But D B is equal (I. *Def.* 15) to D F. Therefore D F is greater than D E, and the point E lies within the circle (III. *Ax.* 1). In the same manner it may be proved that every other point in A B lies within the circle. Therefore the straight line A B lies within the circle. Wherefore, if any two points, &c. Q. E. D.

This demonstration differs from Euclid's, in being direct, and not proceeding by the method of *reductio ad absurdum.* It is preferable too, on account of the greater simplicity of the diagram.

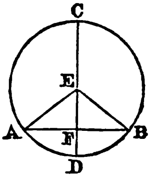
Corollary.—A straight line cannot cut the circumference of a circle in more points than two.

PROP. III. THEOREM.

If in a circle, one straight line passing through the centre of a circle bisect another within it, which does not pass through the centre, the latter is cut at right angles: and conversely, if it be cut at right angles it is bisected.

Let A B C be a circle; and let C D, a straight line passing through the centre, bisect any straight line A B, which does not pass through the the centre, in the point F. The straight line A B is cut at right angles.

Take E the centre of the circle (III. 1), and join E A and E B.



Because A F is equal to F B (*Hyp.*), and F E common to the two triangles A F E, B F E, the two sides A F, F E, in the one are equal to the two sides B F, F E, in the other, each to each. But the base E A is equal (I. *Def.* 15) to the base E B. Therefore the angle A F E is equal (I. 8) to the angle B F E. But when one straight line standing upon another makes the adjacent angles equal (I. *Def.* 10) to one another, each of them is a right angle. Therefore each of the angles A F E, BFE, is a right angle. Wherefore the straight line CD, passing through the centre and bisecting another A B that does not pass through the centre, cuts it at right angles.

Next, let C D cut A B at right angles. The straight line A B is *bisected at F.*

Next only two straight lines drawn to the circumference from F, can be equal to each other, one being on each side of AD.

At the point E in the straight line EF make the angle FEH equal to the angle FEG. Join FH. FH is the only straight line that can be drawn to the circumference equal to FG.

Because GE is equal (I. *Def.* 15) to EH, and EF common to the two triangles GEF, HEF; the two sides GE and EF are equal to the two sides HE and EF, each to each. But the angle GEF is equal (*Const.*) to the angle HEF. Therefore the base FG is equal (I. 4) to the base FH. And no other straight line but FH, can be drawn from F to the circumference equal to FG. For, if possible, let FK be equal to FG. Because FK is equal to FG, and FG to FH, therefore FK is equal (I. *Ax.* 1) to FH; that is, a line nearer to that which passes through the centre, is equal to one more remote; which has been proved to be impossible. Therefore, if any point be taken, &c. Q. E. D.

Corollary 1.—If two chords of a circle intersect each other and make equal angles with the diameter passing through their point of intersection, they are equal.

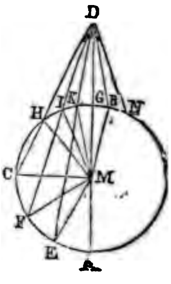
Corollary 2.—If two chords of a circle intersect each other and make unequal angles with the diameter passing through the point of their intersection; that which makes the least angle with the diameter is the greatest.

DEFINITION.—An arc of a circle is said to be *concave* towards a point without it, when all the straight lines, drawn from that point meet the *hollow part* or *inside* of the arc; and it is said to be *convex* towards a point without it, when all the straight lines, drawn from that point, meet the *round part* or *outside* of the arc. Straight lines drawn from any point without a circle to touch the circumference will determine the two points which, being joined, will divide the circumference into the *concave* and *convex* parts.

PROP. VIII. THEOREM.

If from any point without a circle, straight lines be drawn to the circumference; of those which fall upon the concave part of the circumference, the greatest is that which passes through the centre; and of the rest, that which is nearer to the greatest is greater than the more remote; but of those which fall upon the convex part of the circumference, the least is that which produced passes through the centre; and of the rest, that which is nearer to the least is less than the more remote: and only two of either set of straight lines are equal, one being on each side of the diameter.

Let ABC be a circle, and D any point without it. Let the straight lines DA, DE, DF, DC be drawn to the circumference. Of those which fall upon AEFC the concave part of the circumference, DA which passes through the centre, is the greatest; of the rest, DE is greater than DF, and DF than DC. But of those which fall upon HIKG, the convex part of the circumference, DG is the least; of the rest, DK is less than DI, and DI than DH.



Take M the centre of the circle ABC (III. 1), and join ME, MF, MC, MK, MI, and MH.

Because AM is equal to ME (I. *Def.* 15). To each of these equals, add MD. Therefore AD is is equal (I. *Ax.* 2) to EM and MD. But EM and MD are greater than ED (I. 20). Therefore also AD is greater than ED. Again, because ME is equal to MF, and MD common to the two triangles EMD, FMD; the two sides EM and MD are equal to the two sides FM and MD, each to each. But the angle EMD is greater than (I. *Ax.* 9) the angle FMD. Therefore the base ED is greater than (I. 24) the base FD. In like manner, it may be shown that FD is greater than CD. Therefore, DA is the greatest; DE is greater than DF, and DF greater than DC.

Because MK and KD are greater than MD (I. 20) and MK is equal (I. *Def.* 15) to MG. Therefore the remainder KD is greater than (I. *Ax.* 5) the remainder GD; that is, GD is less than KD. Again, because MI is equal to MK, and MD common to the two triangles MID, MKD, the two sides IM and MD are equal to the two sides KM and MD, each to each. But the angle IMD is greater than the angle KMD. Therefore, the base ID is greater than (I. 24) the base KD; that is, KD is less than ID. In like manner it may be shown, that DI, is less than DH. Therefore, DG is the least; DK is less than DI, and DI less than DH.

Lastly, at the point M, in the straight line MD, make the angle DMB equal to the angle DMK (I. 23), and join DB.

Because MK is equal to MB, and MD common to the triangles KMD, BMD, the two sides KM and MD are equal to the two sides BM and MD, each to each. But the angle KMD is equal (*Const.*) to the angle BMD. Therefore the base DK is equal (I. 4) to the base DB. And no straight line drawn from D to the circumference, but DB, can be equal to DK: for, if possible, let DN be equal to DK. Because DN is equal to DK, and it has been shown that DB is equal to DK. Therefore DB is equal (I. *Ax.* 1) to DN; that is, a line nearer to the least is equal to one more remote, which has been proved to be impossible. If, therefore, any point, &c. Q. E. D.

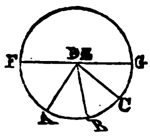
Corollary 1.—**If two secants of a circle, drawn from the same point without it make equal angles with a secant passing through that point and the centre, they are equal.**

Corollary 2.—**If two secants of a circle, drawn from the same point without it, make unequal angles with a secant passing through that point and the centre; that which makes the least angle with it, is the greatest.**

PROP. IX. THEOREM.

If from a point within a circle, more than two equal straight lines can be drawn to the circumference, that point is the centre of the circle.

Let D be a point within the circle ABC, from which the equal straight lines DA, DB, and DC are drawn to the circumference. The point D is the centre of the circle. For if the point D be not the centre, let the point E be the centre. Join DE, and produce it to meet the circumference in the points F and G (I. *Def.* 17).



Because FG is a diameter of the circle ABC, and from the point D, which is not the centre, straight lines DG, DC, DB, and DA are drawn to the circumference. Therefore DG is the greatest, DC is *greater than DB,* and DB greater than DA (III. 7). But they are likewise equal (*Hyp.*); which is impossible. Therefore E is not the centre of the circle A B C. In like manner it may be demonstrated, that no other point but D is the centre. Therefore D is the centre of the circle A B C. Wherefore, if a point be taken, &c. Q. E. D.

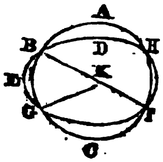
The preceding demonstration depends on the supposition that the three straight lines DA, DB, and DC are all on one side of the diameter FG. But the point E might be so chosen, that DC and DB should be on opposite sides of the diameter, and then DC might be equal to DB instead of being greater than it. There is consequently a defect in this demonstration. The following demonstration, which is free from this defect, is generally adopted instead of it: Because, from any point which is not the centre, *only* two equal straight lines (III. 7) can be drawn to the circumference. Therefore a point from which more than two equal straight lines can be drawn to the circumference, is the centre. Wherefore the point D is the centre of the circle ABC.

Exercise.—Give a direct demonstration of this proposition, founded on the corollary to Prop. III. of this Book.

PROP. X. THEOREM.

If two circles intersect one another, the circumference of the one cannot cut that of the other in more than two points.

Let ABC and DEF, be two circles which intersect one another. Their circumferences cannot cut each other in more than two points.



If it be possible, let the circumference of the circle FAB cut the circumference of the circle DEF in more than two points,—viz., in B, G, and F. Take the centre K of the circle ABC (III. 3), and join KB, KG, and KF.

Because K is the centre of the circle ABC. The straight lines KB, KG and KF are all equal (I. *Def.* 15) to each other. Because from the point K, within the circle DEF, more than two equal straight lines KB, KG, and KF are drawn to the circumference DEF. Therefore the point K is the centre (III. 9) of the circle DEF. But K is also (*Const.*) the centre of the circle ABC. Therefore the same point K is the centre of the two circles ABC and DEF that cut one another; which (III. 5) is impossible. Therefore, one circumference of a circle cannot cut another in more than two points. Q. E. D.

Exercise.—If two circles intersect one another, the straight line which joins their centres, bisects their common chord at right angles.

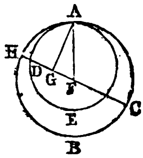
PROP. XI. THEOREM.

If one circle touch another internally in any point, the straight line which joins their centres being produced passes through that point.

Let the circle ADE touch the circle ABC internally in the point A. The straight line which joins their centres being produced passes through the point A.

For if the straight line joining their centres do not pass through the point A, let it pass otherwise, if possible, as FGDH; let F be the centre of the circle ABC, and G the centre of the circle ADE. Join AF and AG.

Because two sides of a triangle A G F are together greater than (I. 20) the third side. Therefore F G and G A are greater than F A. But F A is equal (I. *Def.* 15) to F H. Therefore F G and G A are greater than F H. From these unequals, take away the common part F G. Therefore the remainder A G is greater than (I. *Ax.* 5) the remainder G H. But A G is equal (I. *Def.* 15) to G D. Therefore G D is greater than G H; that is, a part is greater than the whole; which is impossible. Therefore the straight line which joins the centres of the circles A D E and A B C, being produced, cannot pass otherwise than through the point A; that is, it must pass through the point A. Therefore, if one circle, &c. Q. E. D.

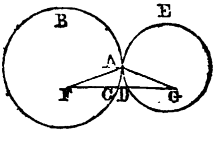


The demonstration of this proposition might be rendered clearer by changing the enunciation as follows:—If one circle touch another internally in any point, that point and the centres of the two circles are in the same straight line. This will form an exercise for the student.

PROP. XII. THEOREM.

If two circles touch each other externally in any point, the straight line which joins their centres shall pass through that point.

Let the two circles A B C, A D E touch each other externally in the point A. The straight line which joins their centres passes through the point of contact A.



For if the straight line joining their centres do not pass through the point A, let it pass otherwise, if possible, as F C D G; let F be the centre of the circle B A C, and G the centre of the circle E A D; and join F A and A G.

Because F is the centre of the circle A B C (I. *Def.* 15), F A is equal to F C. Because G is the centre of the circle A D E, G A is equal to G D. Therefore F A and A G are together equal (I. *Ax.* 2) to F C and D G; and the whole F G is greater (I. *Ax.* 9) than F A and A G. But F A G is a triangle, and F G is also less than F A and A G (I. 20); which is impossible. Therefore the straight line which joins the centres of the circles B A C and E A D cannot pass otherwise than through the point A; that is, it must pass through the point A. Therefore, if two circles, &c. Q. E. D.

The demonstration of this proposition might be improved like the preceding one, by making a similar change on its enunciation as follows:—If two circles touch each other externally in any point, that point and the centres of the two circles are in the same straight line. This will form another exercise for the student.

PROP. XIII. THEOREM.

One circle cannot touch another in more points than one, either internally or externally.

Let the circle E B F touch the circle A B C internally in the point B. E B F cannot touch A B C in any other point.

[illegible]

[illegible] Therefore one circle [illegible] in more points [illegible]

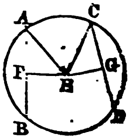
[illegible] proposition might be abridged thus:—If one circle [illegible] points, the straight line which joins the points of contact [illegible] Therefore the centres of the circles are both [illegible] common chord at right angles (III. 1 Cor.). [illegible] passes also through the points of [illegible] Therefore the straight line in which their centres are [illegible] at right angles, and at the same time passes through [illegible] which is impossible. Therefore, &c.

PROP. XIV. THEOREM.

Equal straight lines in a circle are equally distant from the centre; and conversely, straight lines equally distant from the centre, are equal to one another.

Let the straight lines AB and CD, in the circle BACD, be equal to one another. They are equally distant from the centre.

Take E the centre of the circle ABDC (III. 1), and from E draw EF and EG perpendiculars to AB and CD respectively. Join EA and EC.



Because the straight line EF, passing through the centre, cuts the straight line AB, which does not pass through the centre, at right angles (III. 3), AB is bisected at F. Therefore AF is equal to FB, and AB double of A F. For the same reason, C D is double of C G. But A B is equal (*Hyp.*) to C D. Therefore A F is equal (I. *Ax.* 7) to C G, and their squares are equal. Because A E is equal to E C (I. *Def.* 15), the square of A E is equal to the square of E C. But the squares of A F and F E are equal (I. 47) to the square of A E. Also, the squares of E G and G C are equal (I. 47) to the square of E C. Therefore the squares of A F and F E are equal (I. *Ax.* 1) to the squares of C G and G E. From these equals, take the squares of A F and C G, which were shown to be equal. Therefore the remaining square of E F is equal (I. *Ax.* 3) to the remaining square of E G, and the straight line E F to the straight line E G. But straight lines in a circle are said to be equally distant from the centre, when the perpendiculars drawn to them from the centre are equal (III. *Def.* 4). Therefore A B and C D are equally distant from the centre.

Next, let the straight lines A B and C D be equally distant from the centre (III. *Def.* 4); that is, let F E be equal to E G. The straight lines A B and C D are equal.

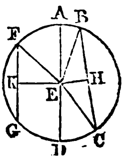
For, the same construction being made, it may, as before, be demonstrated, that A B is double of A F, and C D double of C G, that the squares of F E and E G are equal, and that the squares of E F and F A are equal to the squares of E G and G C. From these equals, take the squares of F E and E G, which are equal. Therefore the remaining square of A F is equal (I. *Ax.* 3) to the remaining square of C G; and the straight line A F to the straight line C G. But A B was shown to be double of A F, and C D double of C G. Therefore A B is equal (I. *Ax.* 6) to C D. Therefore equal straight lines, &c. Q. E. D.

Exercise.—If any number of equal chords in a circle, be bisected, one circle passes through all the points of bisection and touches them at these points,

PROP. XV. THEOREM.

The diameter is the greatest straight line in a circle; and, of all others, that which is nearer to the centre is greater than one more remote: and the greater is nearer to the centre than the less.

Let A B C D be a circle, of which the diameter is A D, and the centre E; and let B C be nearer to the centre than F G. The diameter A D is greater than any other straight line B C, and B C is greater than F G.



From the centre E draw E H and E K perpendiculars to B C and F G (I. 12), respectively, and join E B, E C, and E F.

Because A E is equal (I. *Def.* 15) to E B, and E D to E C. Therefore A D is equal (I. *Ax.* 2) to E B and E C; but E B and E C are greater (I. 20) than B C. Therefore also A D is greater than B C. Because B C is nearer (*Hyp.*) to the centre than F G, E H is less (III. *Def.* 5) than E K. Therefore the square of E H is less than the square of E K. But, it may be shown as in the preceding proposition, that B C is double of B H, and F G double of F K, and that the squares of E H and H B are equal to the squares of E K and K F. But the square of E H is less than the square of E K. Therefore the square of B H is greater than the square of F K, and the straight line B H greater than *the straight line* F K. Therefore also, B C is greater than F G.

Next, let B C be greater than F G. The greater B C is nearer to the centre than the less F G; that is, E H is less than E K.

For let the same construction be made. Because B C is greater than F G (III. *Def.* 5), B H is great than K F, and the square of B H is greater than the square of F K. But, as before, the squares of B H and H E are equal to the squares of F K and K E. Therefore the square of E H is less than the square of E K, and the straight line E H less than E K. Therefore B C is nearer (III. *Def.* 5) to the centre F G. Wherefore the diameter, &c. Q. E. D.

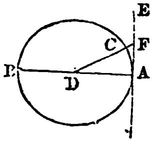
This proposition might be proved by producing E H, cutting off the line thus produced a part equal to E K, and through the point of section, drawing a chord parallel to B C, meeting the circumference. By the preceding proposition this chord would be equal to F G. It could then be proved (I. 24) that B C is greater than this chord. Therefore B C is greater than F G. To prove the remaining part of the proposition in the same way, will be an exercise for the student.

Exercise.—The shortest chord which can be drawn through a given point in a circle is that which is perpendicular to the diameter passing through that point.

PROP. XVI. THEOREM.

If a straight line pass through one of the extremities of the diameter of a circle, at right angles to the diameter, it touches the circumference; but if it pass through the same point not at right angles to the diameter, it cuts the circumference.

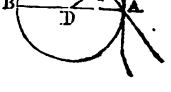
Let A B C be a circle, A B its diameter, and D the centre. Let the straight line A E pass through the point A, the extremity of the diameter, at right angles to A B. The straight line A E touches the circle A B C at the point A; that is, A E lies wholly without the circumference, except at the point A.



In A E take any point F, and join D F. Let D F meet the circumference at C.

Because, in the triangle D A F, the angle D A F (*Hyp.*) is a right angle. Therefore the angle D A F is greater than (I. 17) the angle D F A, and D F is greater than D A (I. 19). But D A is equal to D C (I. *Def.* 15). Therefore D F is greater than D C, and the point F (III. *Ax.* 1) is without the circumference. In the same manner it may be shown that any point in A E, but the point A, is without the circumference. Therefore A E lies wholly without the circumference, except at the point A. Wherefore A E touches the circumference at the point A.

Next, let the straight line A C pass through the point A, the extremity of the diameter, not at right angles to A B. The straight line A C cuts the circumference of the circle A B C.



From the centre D draw D F (I. 12) at right angles to A C, and let it meet the circumference at G.

Because, in the triangle D A F, the angle D F A is a right angle (*Const.*), the angle D A F (I. 17) is less than the angle D F A. Therefore D F is less than D A (I. 19). But D A is equal to D G (I. *Def.*

15). Therefore D F is less than D G, and the point F is within the circumference (III. *Ax.* 1). But the point F is in the straight line A C. Therefore the straight line A C cuts the circumference. Therefore, if a straight line pass, &c. Q. E. D.

The demonstration of the preceding proposition as given by Euclid, besides being indirect, is rather obscure, notwithstanding Dr. Simson's improvements. The above demonstration is preferable, on account of its being direct, and expressed in fewer words. The enunciation and demonstration have also been rendered more full and distinct, by the incorporation of the corollary respecting the tangent, or straight line which touches the circle, usually appended to this proposition.

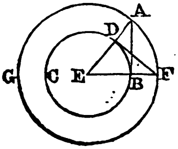
Corollary 1.—If several circles touch each other in the same point, either internally or externally, they have all the same tangent at the point of contact.

Corollary 2.—The tangents at the extremities of the same diameter of a circle are parallel.

PROP. XVII. PROBLEM.

To draw a straight line from a given point, either in the circumference of a given circle, or without it, that shall touch the circumference.

Let B D C be the given circle, D a point in the circumference, and A a point without it. It is required to draw from A or D a straight line that shall touch the circumference.



First, when the point A is without the circumference. Find the centre of the circle B C D, and join A E. From the centre E, at the distance E A, describe the circle A F G. From the point D, draw D F at right angles to E A (I. 11). Join E F and A B. The straight line A B touches the circle B C D at the point B.

Because E is the centre of the circles B C D and A F G, E A is equal to E F (I. *Def.* 15) and E D to E B. Therefore, in the two triangles A E B, F E D, the two sides A E and E B are equal to the two sides F E and E D, each to each; and they contain the angle at E common to the two triangles A E B and F E D. Therefore the base D F is equal to the base A B (I. 4), the triangle F E D to the triangle A E B, and the remaining angles to the remaining angles of the other, each to each. Therefore the angle E B A is equal to the angle E D F. But E D F is a right angle (*Const.*). Therefore E B A is a right angle (I. *Ax.* 1). But a straight line which passes through the extremity of a diameter (or radius), at right angles to it, touches the circle (III. 16). Therefore A B touches the circle; and it is drawn from the given point A.

Next, when the given point D is in the circumference of the circle. Find the centre E as before. Join D E, and draw D F at right angles to D E (I. 11). For the same reason as above D F touches the circle (III. 16), and it is drawn from the given point D. Q. E. F.

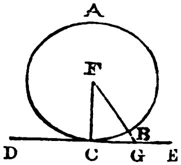
Corollary 1.—From the same point in the circumference only one tangent can be drawn to the circle; but from the same point without the circumference two tangents can be drawn, and these are equal to one another.

Corollary 2.—The chords in a circle which touch a concentric circle are equal to one another, and are each bisected at the point of contact.

PROP. XVIII. THEOREM.

If a straight line touch a circle, the straight line drawn from the centre to the point of contact, is perpendicular to the line touching the circle.

Let the straight line D E touch the circle A B C in the point C, and from F, the centre, let the straight line F C be drawn to the point of contact C. F Ci s perpendicular to D E.



For, if F C be not perpendicular to D E; from the point F draw F G perpendicular to D E and let it meet the circumference in B.

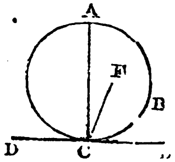
Because F G C is a right angle, G C F is an acute angle (I. 17). But in any triangle, the greater side is opposite (I. 19) to the greater angle. Therefore F C is greater than F G. But F C is equal to F B (I. *Def.* 15). Therefore F B is greater than F G; that is, a part greater than the whole, which is impossible. Therefore F G is not perpendicular to D E. In the same manner it may be shown, that no other straight line can be perpendicular to D E but F C. Therefore F C is perpendicular to DE. Therefore, if a straight line, &c. Q. E. D.

If the enunciation of this proposition be slightly altered in form, it will appear to to be the converse of the first part of Prop. XVI. Thus, if a straight line touch a circle, the diameter which passes through the point of contact is perpendicular to the tangent.

PROP. XIX. THEOREM.

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the tangent, the centre of the circle shall be in that straight line.

Let the straight line D E touch the circle A B C at the point C; and from C, let C A be drawn at right angles to D E. The centre of the circle is in the straight line C A.



For if the centre of the circle be not in C A, let, if possible, F a point out of C A be the centre, and join C F.

Because D E touches the circle A B C at the point C, and F C is drawn from the centre to the point of contact. Therefore F C is perpendicular to D E (III. 18), and F C E is a right angle. But A C E is also (*Hyp.*) a right angle. Therefore the angle F C E is equal (I. *Ax.* 1) to the angle A C E; that is, the less to the greater, which is impossible. Therefore F is nòt the centre of the circle A B C. In the same manner it may be shown, that no other point out of C A, is the centre. Therefore the centre of the circle is in C A. Therefore, if a straight line, &c. Q. E. D.

If this proposition be slightly altered in its enunciation it will appear to be another form of the converse to the first part of Prop. XVI. If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the tangent, it is a diameter. This suggests, theoretically speaking, an easy mode of finding the centre of a given circle.

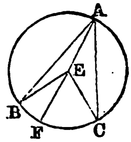
***Exercise.*—Through a given point within or without a given circle, to draw a**

chord equal to a given straight line. For either point, the given straight line must not be greater than the diameter of the circle. For the internal point, the given straight line must not be less than the chord drawn at right angles to the diameter passing through that point.

PROP. XX. THEOREM.

The angle at the centre of a circle is double of the angle at the circumference standing upon the same arc, that is, upon the same part of the circumference.

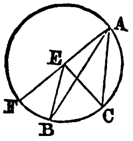
Let ABC be a circle, BEC an angle at the centre, and BAC an angle at the circumference, standing upon the same arc or part of the circumference BC. The angle BEC is double of the angle BAC.



Join AE, and produce it to F. First, let the centre of the circle be within the angle BAC.

Because EA is equal to EB, the angle EAB is equal (I. 5) to the angle EBA. Therefore the two angles EAB and EBA are double of the angle EAB. But the angle BEF is equal (I. 32) to the two angles EAB and EBA. Therefore also the angle BEF is double of the angle EAB. For the same reason, the angle FEC is double of the angle EAC. Therefore the whole angle BEC is double (III. *Ax.* 2) of the whole angle BAC.

Next, let the centre of the circle be without the angle BAC.



It may be demonstrated, as in the preceding case, that the angle FEC is double of the angle FAC, and that FEB, a part of the first, is double of FAB, a part of the other. Therefore the remaining angle BEC is double (III. *Ax.* 3) of the remaining angle BAC. Therefore the angle at the centre, &c. Q. E. D.

In this proposition, there ought to be three cases; viz., 1. When the centre of the circle is *on* a leg of the angle at the circumference; 2. When it is *within* the legs of that angle; and 3. When it is *without* the legs of that angle. Euclid has only given the two latter cases as above demonstrated; but the demonstration of the first case is included in that of the second (which is Euclid's first), by considering the angle at the centre, as FEC, and the angle at the circumference as FAC.

When the angle at the circumference reaches or exceeds a right angle in magnitude, it is plain that the angle at the centre must, if this proposition be universally true, reach or exceed two right angles. Now the demonstration above given holds equally true in these cases; and it is probable that Euclid considered them as included in it, because he has omitted in the Greek, the second case of the next proposition, which expressly requires this extension of meaning in the present proposition.

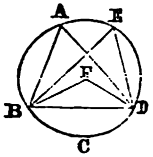
PROP. XXI. THEOREM.

The angles in the same segment of a circle are equal to one another.

Let ABCD be a circle, and BAD and BED angles in the same segment BAED. The angles BAD and BED are equal to one another.

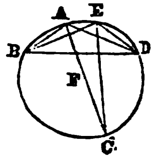
First, let the segment BAED be greater than a semicircle.

Take F, the centre of the circle ABCD (III. 1), and join BF and FD.



Because the angle BFD at the centre, and the angle BAD at the circumference, stand upon the same arc BCD, the angle BFD is double (III. 20) of the angle BAD. For the same reason, the angle BFD is double of the angle BED. Therefore the angle BAD is equal (I. *Ax.* 7) to the angle BED.

Next, let the segment BAED be not greater than a semicircle.



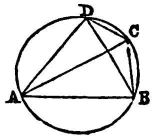
Join AF, produce it to C, and join CE. Because the segment BADC is greater than a semicircle; the angles BAC and BEC are equal, by the first case. Because CBED is greater than a semicircle, the angles CAD and CED are also equal, by the first case. Therefore the whole angle BAD is equal (I. *Ax.* 2) to the whole angle BED. Wherefore the angles in the same segment, &c. Q. E. D.

Corollary.—The vertices of any number of equal angles, whose legs all pass through the same two points, and on the same side of the straight line which joins them, are in the circumference of a circle, of which that straight line is a chord.

PROP. XXII. THEOREM.

The opposite angles of any quadrilateral figure inscribed in a circle, are together equal to two right angles.

Let ABCD be a quadrilateral figure in the circle ABCD. Any two of its opposite angles are together equal to two right angles.



Join AC and BD. The angle CAB is equal to the angle CDB (III. 21), because they are in the same segment CDAB. And the angle ACB is equal to the angle ADB, because they are in the same segment ADCB. Therefore the two angles CAB and ACB are together equal (I. *Ax.* 2) to the whole angle ADC. To each of these equals, add the angle ABC. Therefore the three angles ABC, CAB, and BCA are equal to the two angles ABC and ADC (I. *Ax.* 2). But the three angles ABC, CAB, and BCA are equal (I. 32) to two right angles. Therefore the two angles ABC and ADC are equal (I. *Ax.* 1) to two right angles. In the same manner it may be shown that the two angles BAD and DCB are equal to two right angles. Therefore the opposite angles, &c. Q. E. D.

This proposition may be demonstrated by reference to Prop. XX. in its extended meaning. Thus, the two angles at the circumference ABC and ADC, with the angles at the centre, standing on the same arcs (though not drawn in the figure), stand on the whole circumference. But the angles at the centre standing on the whole circumference are equal to four right angles, and they are also double of the angles ABC and ADC. Therefore the angles ABC and ADC are equal to two right angles.

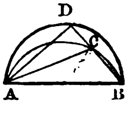
Corollary.—If the opposite angles of a quadrilateral inscribed in a circle be equal to one another, they are both right angles.

Exercise.—If the opposite angles of a quadrilateral be supplementary to each other a circle may be described about it.

PROP. XXIII. THEOREM.

Upon the same straight line, and upon the same side of it, there cannot be two similar segments of circles, not coinciding with one another.

Upon the same straight line AB, and upon the same side of it, let there be, if possible, two similar segments of circles, ACB and ADB. not coinciding with one another.



Because the circumference ACB cuts the circumference ADB in the two points A and B, they cannot cut one another (III. 10) in any other point. Therefore one of the segments must fall within the other. Let ACB fall within ADB. Draw the straight line BCD, and join CA and DA.

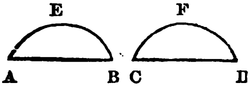
Because the segment ACB is similar (*Hyp.*) to the segment ADB, and similar segments of circles contain (III. *Def.* 11) equal angles. Therefore the angle ACB is equal to the angle ADB; that is, the exterior angle of the triangle ACD is equal to its interior angle, which is impossible (I. 16). Therefore upon the same straight line, and on the same side of it there cannot be two similar segments of circles which do not coincide. Q. E. D.

Corollary.—**If there be two segments of circles on the same base or chord, the larger segment contains the smaller angle.**

PROP. XXIV. THEOREM.

Similar segments of circles upon equal straight lines, are equal to one another, and have equal arcs.

Let AEB and CFD be similar segments of circles upon the equal straight lines AB and CD. The segment AEB is equal to the segment CFD, and the arc AEB to the arc CFD.



For if the segment AEB be applied to the segment CFD, so that the point A may be on the point C, and the straight line AB upon the straight line CD. The point B shall coincide with the point D, because AB is equal to CD. Therefore, the straight line AB coinciding with the straight line CD, the segment AEB must coincide (III. 23) with the segment CFD, because they are similar. Therefore the segment AEB is equal (I. *Ax.* 8) to the segment CFD, and the arc AEB to the arc CFD. Wherefore similar segments, &c. Q. E. D.

Corollary.—**Sectors, of which the arcs have equal chords, are equal.**

PROP. XXV. PROBLEM.

A segment of a circle being given, to describe the circle of which it is the segment.

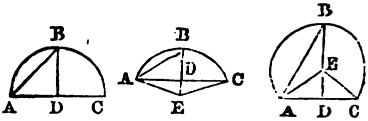
Let ABC be the given segment of a circle. It is required to describe the circle of which it is the segment.

Bisect A C in D (I. 10), from the point D draw D B at right angles to A C (I. 11) and join A B.

First, let the angles A B D and B A D be equal to one another.

Because the straight line B D is equal to D A (I. 6), and also (I. *Ax.* 1) to D C, the three straight lines D A, D B, and D C are all equal. Therefore D is the centre (III. 9) of the circle. From the centre D, at the distance of any of the three straight lines D A, D B, or D C, describe a circle, and it shall pass through the other points. Therefore the circle of which A B C is a segment is described.

Next, let the angles A B D and B A D be unequal to one another.



At the point A, in the straight line A B, make the angle B A E equal (I. 23) to the angle A B D. Produce B D, if necessary, to meet A E in E, and join E C.

Because the angle A B E is equal to the angle B A E, the straight line B E is equal (I. 6) to the straight line E A. Because A D is equal to D C, and D E common to the triangles A D E and C D E, the two sides A D and D E, are equal to the two sides C D and D E, each to each. But the angle A D E is equal to the angle C D E, for each of them (*Const.*) is a right angle. Therefore the base A E is equal (I. 4) to the base E C. But A E was shown to be equal to E B. Therefore the three straight lines A E, E B, and E C are equal (I. *Ax.* 1) to one another, and E is the centre (III. 9) of the circle. From the centre E, at the distance of any of the three straight lines A E, E B, or E C describe a circle, and it shall pass through the other points. Therefore the circle, of which A B C is a segment, is described. Wherefore, a segment of a circle being given, the circle is described of which it is a segment. Q. E. F.

COROLLARY.—In the first case, because the centre D is in A C, the segment A B C is a semicircle. In the second case, if the angle A B D be greater than the angle B A D, the centre E falls without the segment A B C, and it is therefore less than a semicircle. But if the angle A B D be less than B A D, the centre E falls within the segment A B C, and it is therefore greater than a semicircle.

This problem may be most easily solved by one construction as follows:—Draw any two chords A B and A C; bisect them, and from the points of bisection draw straight lines at right angles to these chords, producing them till they meet. The point where they meet is the centre of the circle. Because (III. 1. *Cor.*) the centre of the circle is in each of these perpendiculars, it must be at the point where they meet. This problem is, in fact, the same as that given in Prop. I.; viz., To find the centre of a circle, of which the whole or part only of the circumference is given.

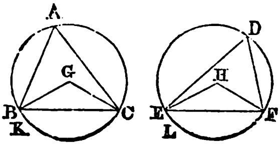
PROP. XXVI. THEOREM.

In equal circles, equal angles stand upon equal arcs, whether they be at the centres or circumferences.

Let A B C and D E F be equal circles, and let the angles B G C and

E H F at their centres, and B A C and E D F at their circumferences, be equal to each other. The arc B K C is equal to the arc E L F.

Join B C and E F.



Because the circles A B C and D E F are equal, the straight lines drawn from their centres (III. *Def.* 1) are equal; therefore the two sides B G and G C, are equal to the two E H and H F, each to each. But the angle at G is equal (*Hyp.*) to the angle at H. Therefore the base B C is equal (I. 4) to the base E F. Because the angle at A is equal (*Hyp.*) to the angle at D, the segment B A C is similar (III. *Def.* 11) to the segment E D F, and they are upon equal straight lines B C and E F. But similar segments of circles upon equal straight lines, are equal (III. 24) to one another. Therefore the segment B A C is equal to the segment E D F, and the arc B A C to the arc E D F. But the whole circumference A B C is equal (*Hyp.*) to the whole circumference D E F. Therefore the remaining segment B K C is equal (I. *Ax.* 3) to the remaining segment E L F, and the arc B K C to the arc E L F. Wherefore, in equal circles, &c. Q. E. D.

Corollary.—**Equal angles either at the centre or the circumference of the same circle, stand upon equal arcs.**

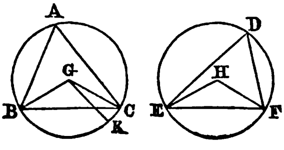
Exercise.—**Parallel chords in a circle intercept (take between them) equal arcs.**

PROP. XXVII. THEOREM.

In equal circles, the angles which stand upon equal arcs, are equal to one another, whether they be at the centres or circumferences.

Let A B C and D E F be equal circles, and let the angles B G C and E H F at their centres, and B A C and E D F at their circumferences, stand upon the equal arcs B C and E F. The angle B G C is equal to the angle E H F, and the angle B A C to the angle E D F.

If the angle B G C be not equal to the angle E H F, one of them must be greater than the other. Let the angle B G C be the greater, and at the point G, in the straight line B G, make the angle B G K equal (I. 23) to the angle E H F.



Because the angle B G K is equal to the angle E H F, and equal angles stand (III. 26) upon equal arcs. Therefore the arc B K is equal to the arc E F. But the arc E F is equal (*Hyp.*) to the arc B C. Therefore also B K is equal (I. *Ax.* 1) to B C, the less to the greater, which is impossible. Wherefore the angle B G C is not unequal to the angle E H F; that is, the angle B G C is equal to the angle E H F. But the angle at A is half of the angle B G C (III. 20) and the angle at D half of the angle E H F. Therefore the angle at A is equal (I. *Ax.* 7) to the angle at D. Wherefore, in equal circles, &c. Q. E. D.

Corollary.—**The angles either at the centre or the circumference of the same circle, which stand upon equal arcs, are equal.**

Exercise 1.—**The chords which intercept equal arcs of a circle, are parallel.**

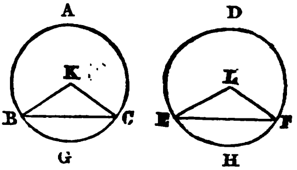
Exercise 2.—To draw a straight line through a given point that shall be parallel to a given straight line; by means of the preceding corollary.

PROP. XXVIII. THEOREM.

In equal circles, equal straight lines cut off equal arcs, the greater equal to the greater, and the less to the less.

Let A B C and D E F be equal circles, and B C and E F equal straight lines in them, which cut off the two greater arcs B A C and E D F, and the two less arcs B G C and E H F. The greater arc B A C is equal to the greater arc E D F, and the less arc B G C to the less arc E H F.

Take K and L, the centres of the circles (III. 1), and join B K, K C, E L, and L F.



Because the circles A B C and D E F are equal, the straight lines drawn from their centres (III. *Def.* 1) are equal. Therefore B K and K C are equal to E L and L F, each to each. But the base B C is equal (*Hyp.*) to the base E F. Therefore the angle B K C is equal (I. 8) to the angle E L F. Because equal angles stand upon equal arcs (III. 26) the arc B G C is equal to the arc E H F. But the whole circumference A B C is equal (*Hyp.*) to the whole circumference E D F. Therefore the remaining arc B A C, is equal (I. *Ax.* 3) to the remaining arc E D F. Therefore, in equal circles, &c. Q. E. D.

In this and the following propositions, it is assumed by Euclid, that the straight lines or chords are less than the diameter. Dr. Thomson, in his edition, adapts the enunciation and demonstration of both to the case of the diameter also; but this seems to be an unnecessary refinement, as the diameter evidently bisects the circles, making the arcs equal, and conversely.

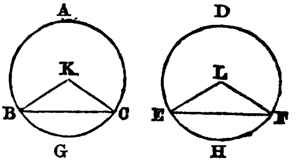
Corollary.—In the same circle, equal chords cut off equal arcs, the greater equal to the greater, and the less to the less.

PROP. XXIX. THEOREM.

In equal circles, equal arcs are subtended by equal straight lines.

Let A B C and D E F be equal circles, and let the equal arcs B G C and E H F be subtended by the straight lines B C and E F. The straight line B C is equal to the straight line E F.

Take K and L (III. 1) the centres of the circles, and join B K, K C, E L, and L F.



Because the arc B G C is equal to the arc E H F, the angle B K C is equal (III. 27) to the angle E L F. Because the circles A B C and D E F are equal, the straight lines from their centres are equal (III. *Def.* 1). Therefore B K and K C are equal E L and L F, each to each. But the angle B K C is equal to the angle

E L F. Therefore the base B C is equal (I. 4) to the base E F. Therefore, in equal circles, &c. Q. E. D.

Corollary.—In the same circle, equal arcs are subtended by equal straight lines.

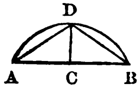
PROP. XXX. PROBLEM.

To bisect a given arc; that is, to divide it into two equal parts.

Let A D B be the given arc. It is required to bisect it.

Join A B, and bisect (I. 10) it at C. From the point C, draw C D at right angles (I. 11) to A B. The arc A D B is bisected at the point D.

Join A D and D B. Because A C is equal to C B, and C D common to the triangles A C D and B C D, the two sides A C and C D are equal to the two sides B C and C D, each to each. But the angle A C D is equal to the angle B C D, because each of them is a right angle. Therefore the base A D is equal (I. 4) to the base B D. But equal straight lines cut off equal (III. 28) arcs, the greater equal to the greater, and the less to the less; and A D and D B are each of them less than a semicircle, because D C (III. 1. *Cor.*) passes through the centre. Therefore the arc A D is equal to the arc D B. Therefore the given arc is bisected at D. Q. E. F.



Corollary.—The perpendicular which bisects any chord of a circle, bisects also the arc which it cuts off.

The *trisection* of any arc of a circle on elementary principles, is as impossible as the trisection of an angle.

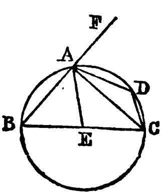
PROP. XXXI. THEOREM.

In a circle, the angle in a semicircle is a right angle; the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Let A B C D be a circle, of which the diameter is B C, and the centre E; and, let any straight line C A divide the circle into the segments A B C and A D C. The angle in the semicircle B A C is a right angle; the angle in the segment A B C, which is greater than a semicircle is less than a right angle; and the angle in the segment A D C, which is less than a semicircle, is greater than a right angle.

Take any point D in the arc A D C, and join A B, A D, D C, and A C. Also join A E, and produce B A to F.

Because B E is equal (I. *Def.* 15) to E A, the angle E A B is equal (I. 5) to the angle E B A. Because A E is equal to E C, the angle E A C is equal to the angle E C A. Therefore the whole angle B A C is equal (I. *Ax.* 2) to the two angles A B C and A C B. But F A C, the exterior angle of the triangle A B C, is equal (I. 32) to the two angles A B C and A C B. Therefore the angle B A C is equal (I. *Ax.* 1) to the angle F A C, and each of them (I. Def. 10) is a right angle. Therefore the angle B A C in a semicircle is a right angle. Because the two angles A B C and B A C of the triangle A B C are



together less than two right angles (I. 17), and BAC has been proved to be a right angle. Therefore A B C is less than a right angle. Wherefore the angle in a segment ABC greater than a semicircle, is less than a right angle. Because ABCD is a quadrilateral figure inscribed in a circle, any two of its opposite angles are equal (III. 22) to two right angles. Therefore the angles A B C and ADC, are equal to two right angles. But A B C has been proved to be less than a right angle. Therefore the angle ADC is greater than a right angle. Wherefore the angle in a segment A D C less than a semicircle is greater than a right angle. Therefore, in a circle, &c. Q. E. D.

Cor. From this it is manifest, that if one angle of a triangle be equal to the other two, it is a right angle: because the angle adjacent to it is equal (I. 32) to the same two; and when the adjacent angles are equal (I. *Def.* 10) they are right angles.

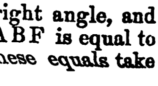
Dr. Thomson, in his edition, simplifies the demonstration of this proposition, by the application of Prop. XX., after the following manner:—The angle BAC at the circumference is half the two angles at the centre formed by producing A E. But these angles are equal to two right angles. Therefore B A C is a right angle; and it is the angle in a semicircle B A C. The angle B A D is greater than the angle B A C (I. *Ax.* 9). Therefore the angle BAD is greater than a right angle; and it is the angle in a segment BAD, less than a semicircle. In like manner, by drawing a straight line from the point A, within the angle B A C, it may be shown that the angle in a segment greater than a semicircle is less than a right angle.

Exercise.—To draw a perpendicular to a straight line from one of its extremities, without producing it, by means of the first part of this proposition.

PROP. XXXII. THEOREM.

If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle; the angles which this straight line makes with the tangent are equal to the angles in the alternate segments of the circle.

Let the straight line EF touch the circle ABCD at the point B; and from the point B, let the straight line BD be drawn, cutting the circle, and dividing it into the two segments DCB and DAB, of which DCB is less, and DAB greater than a semicircle. The angles which BD makes with the tangent EF are equal to the angles in the alternate segments of the circle; that is, the angle DBF is equal to the angle in the segment D A B, and the angle D B E to the angle in the segment DCB.



From the point B, draw BA at right angles (I. 11) to E F, take any point C in the arc DCB, and join AD, DC, and CB.

Because the straight line EF touches the circumference of the circle ABCD at the point B, and BA is drawn at right angles to the tangent from the point of contact B, the centre of the circle is (III. 19) in B A. Therefore the angle A D B in a semicircle (III. 31) is a right angle. Because the other two angles BAD and ABD, in the triangle ADB, are equal to (I. 32) a right angle, and ABF is (*Const.*) a right angle. Therefore the angle ABF is equal to the two angles BAD and ABD (I. *Ax.* 1). From these equals take away the common angle ABD. Therefore the remaining angle DBF is equal to the remaining angle BAD (I. *Ax.* 3), which is in the alternate segment of the circle. Because ABCD is a quadrilateral figure in a circle, the opposite angles BAD and BCD are equal (III. 22) to two right angles. But the two angles DBF and DBE are equal (I. 13) to two right angles. Therefore the two angles DBF and DBE are equal (I. *Ax.* 1) to the two angles BAD and BCD. But the angle DBF has been proved equal to the angle BAD. Therefore the remaining angle DBE is equal (I. *Ax.* 2) to the remaining angle BCD, which is in the alternate segment of the circle. Wherefore, if a straight line, &c. Q. E. D.

In Dr. Thomson's edition, the case, in which the straight line cutting the circle passes through the centre, is considered; this is so evident, that it appears to be an unnecessary addition.

Corollary 1.—If the angles which any straight line meeting a circle makes with a chord, be equal to the angles in the alternate segments, that straight line touches the circle.

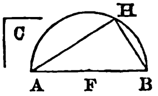
Corollary 2.—Tangents to a circle at the extremities of the same chord are equal, and make equal angles with it on the same side.

Corollary 3.—The chord which joins the points of contact of parallel tangents is a diameter.

PROP. XXXIII. PROBLEM.

Upon a given straight line to describe a segment of a circle, which shall contain an angle equal to a given rectilineal angle.

Let AB be the given straight line, and C the given rectilineal angle. It is required to describe upon the given straight line AB, a segment of a circle, which shall contain an angle equal to the angle C.

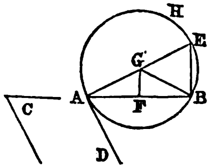


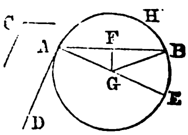
First, let the angle C be a right angle.

Bisect AB in F (I. 10), and from the centre F, at the distance FB, describe the semicircle AHB. Because the segment AHB is a semicircle, the angle AHB is an angle (III. 31), and therefore it is equal to the given angle C.

Next, let the angle C be an oblique angle.

At the point A, in the straight line AB, make the angle BAD equal to the angle C (I. 23); and from the point A, draw AE at right angles (I. 11) to AD. Bisect AB in F (I. 10); and from F, draw FG at right angles (I. 11) to AB. Join GB.





Because AF is equal to FB, and FG common to the triangles AFG and BFG, the two sides AF and FG are equal to the two sides BF and FG, each to each. But the angle AFG is equal (I. Def. 10) to the angle BFG. Therefore the base AG is equal (I. 4) to the base GB. With centre G and distance GA describe the circle AHB, and it shall pass through the point B. The segment AHB contains an angle equal to the given rectilineal angle C.

Because from the point A, the extremity of the diameter AE, the straight line AD is drawn at right angles to AE, the straight line AD touches the circle (III. 16). Because AB is drawn from the point of contact A, cutting the circle, the angle DAB is equal (III. 32) to the angle in the alternate segment AHB. But the angle DAB is equal (*Const.*) to the angle C. Therefore the angle C is equal (I. Ax. 1) to the angle in the segment AHB. Therefore, upon the given straight line AB, the segment of a circle AHB is described, which contains an angle equal to the given angle C. Q. E. F.

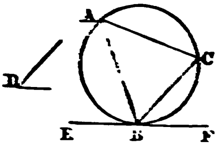
> The construction of this proposition may be varied by making the angle GBA equal to the angle GAB; as this is another mode of finding the centre G.

> *Exercise.*—Given the base, the angle opposite the base, and the perpendicular of a triangle, to construct the triangle.

PROP. XXXIV. PROBLEM.

From a given circle to cut off a segment, which shall contain an angle equal to a given rectilineal angle.

Let ABC be the given circle, and D the given rectilineal angle. It is required to cut off from the circle ABC a segment that shall contain an angle equal to the given angle D.



Draw the straight line EF touching the circle ABC in the point B (III. 17), and at the point B, in the straight line BF, make the angle FBC equal (I. 23) to the angle D. The segment BAC contains an angle equal to the given angle D.

Because the straight line EF touches the circle ABC, and BC is drawn from the point of contact B, the angle FBC is equal (III. 32) to the angle in the alternate segment BAC of the circle. But the angle FBC is equal (*Const.*) to the angle D. Therefore the angle in the segment BAC is equal (I. *Ax.* 1) to the angle D. Wherefore, from the given circle ABC, the segment BAC is cut off, containing an angle equal to the given angle D. Q. E. F.

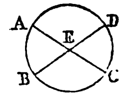
> *Exercise.*—Through a given point to draw a straight line that shall cut off, from a given circle, a segment containing a given angle.

PROP. XXXV. THEOREM.

If in a circle two straight lines cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

In the circle ABCD let the two straight lines AC and BD, cut one another in the point E. The rectangle contained by AE and EC is equal to the rectangle contained by BE and ED.

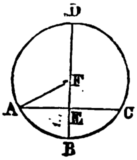
First, let A C and B D pass each through the centre E. Because the segments AE, EC, B E and ED are all equal (I. *Def.* 15). Therefore the rectangle A E.E C, is equal to the rectangle BD.ED.



Secondly, let the one B D pass through the centre, and cut the other A C, which does not pass through the centre, at right angles, in the point E.

Find F, the centre of the circle A B C D, and join A F.

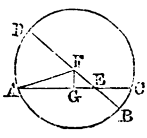
Because B D passing through the centre, cuts the straight line A C, which does not pass through the centre, at right angles in E (III. 3), A E is equal to E C. Because the straight line B D is cut into two equal parts at the point F, and into two unequal parts at the point E, the rectangle BE.ED, together with the square of EF, is equal (II. 5) to the square of F B; that is, to the square of F A, because F A is equal to F B. But the squares of A E and E F, are equal (I. 47) to the square of F A. Therefore the rectangle B E.ED together with the square of EF, is equal (I. *Ax.* 1) to the squares of A E and E F. From these equals take away the common square of E F. Therefore the remaining rectangle B E.E D is equal (I. *Ax.* 3) to the remaining square of A E; that is, to the rectangle A E.E C.



Thirdly, let B D, passing through the centre, cut the other A C, which does not pass through the centre, at E, but not at right angles.

Find F the centre of the circle, join A F, and from F draw F G perpendicular (I. 12) to A C.

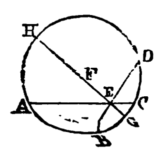
Because A G is equal (III. 3) to G C, the rectangle A E.E C, together with the square of E G, is equal (II. 5) to the square of A G. To each of these equals, add the square of G F. Therefore the rectangle A E.E C, together with the squares of E G and G F, is equal (I. *Ax.* 2) to the squares of A G and G F. But the squares of E G and G F, are equal (I. 47) to the square of E F. And the squares of A G and G F are equal to the square of A F. Therefore the rectangle A E.E C, together with the square of E F, is equal to the square of A F; that is, to the square of F B. But the square of F B is equal (II. 5) to the rectangle B E.E D, together with the square of E F. Therefore the rectangle A E.E C, together with the square of E F, is equal (I. *Ax.* 1) to the rectangle B E.E D, together with the square of E F. From these equals take away the common square of E F. Therefore the remaining rectangle A E.E C, is equal (*Ax.* 3) to the remaining rectangle B E.E D.



Lastly, let neither of the straight lines A C and B D cutting each other, pass through the centre.

Find the centre F (III. 1), and through E, the point of intersection of the straight lines A C and D B, draw the diameter G H.

Because the rectangle A E.E C is equal, by the preceding case, to the rectangle G E.E H. And the rectangle B E.E D is equal to the same rectangle



GE.EH. Therefore the rectangle AE.EC is equal (I. *Ax.* 1) to the rectangle BE.ED. Wherefore, if two straight lines, &c. Q. E. D.

The second case of this proposition is included in the demonstration of Prop. XIV, Book II. A demonstration including all the cases may be derived from Props. IV. and XVI., of Book VI.

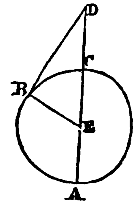
Corollary.—If the rectangles contained by the segments of the diagonals of any quadrilateral figure, be equal, a circle may be described about it.

PROP. XXXVI. THEOREM.

If from any point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, is equal to the square of the tangent.

Let D be any point without the circle ABC, from which the two straight lines DA and DB are drawn, of which DA cuts the circle and DB touches it. The rectangle AD.DC is equal to the square of DB.

First, let DA pass through the centre E.

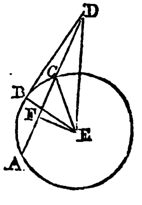


Join EB, and (III. 18) EBD is a right angle.

Because the straight line AC is bisected in E, and produced to D, the rectangle AD.DC, together with the square of EC, is equal (II. 6) to the square of ED. But CE is equal to EB. Therefore the rectangle AD.DC, together with the square of EB, is equal to the square of ED. But the square of ED is equal (I. 47) to the squares of EB and BD. Therefore the rectangle AD.DC, together with the square of EB, is equal (I. *Ax.* 1) to the squares of EB and BD. From these equals, take away the common square of EB. Therefore the remaining rectangle AD.DC is equal (*Ax.* 3) to the square of the tangent DB.

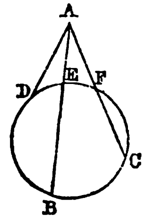
Next let DA not pass through the centre of the circle ABC.

Find E the centre of the circle (III. 1), draw EF perpendicular to AC (I. 12) and join EB, EC, and ED.



Because the straight line EF, passing through the centre, cuts the straight line AC, which does not pass through the centre, at right angles, it bisects AC (III. 3). Therefore AF is equal to FC. Because the straight line AC is bisected in F, and produced to D, the rectangle AD.DC together with the square of FC, is equal (II. 6) to the square of FD. To each of these equals, add the square of FE. Therefore the rectangle AD.DC, together with the squares of CF and FE is equal (I. *Ax.* 2) to the squares of DF and FE. But the square of ED is equal (I. 47) to the squares of DF and FE. Also, the square of EC is equal to the squares of CF and FE. Therefore the rectangle AD.DC, together with the square of EC is equal (I. *Ax.* 1) to the square of ED. But CE is equal to EB. Therefore the rectangle AD.DC, together with the square of EB, is equal to the square of ED. But the squares of EB and BD are equal (I. 47) to the square of ED. Therefore the rectangle AD.DC, together with the square of EB, is equal to the squares of EB and BD. From these equals take away the common square of EB. Therefore the remaining rectangle AD.DC is equal (I. *Ax.* 3) to the square of DB. Wherefore, if from any point, &c. Q. E. D.

Cor.—If from any point without a circle, there be drawn two straight lines cutting it, as AB and AC, the rectangles contained by the whole lines and the parts of them without the circle, are equal to one another; viz., the rectangle BA.AE, to the rectangle CA.AF: for each of them is equal to the square of the straight line AD, which touches the circle.



A demonstration including both cases of this proposition may be derived from Props. IV. and XVII. of Book VI.

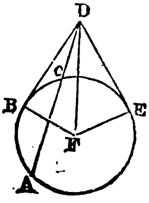
***Exercise* 1.—If two circles cut each other, the straight line joining the points of intersection, if produced, bisects the straight line which touches both circles.**

***Exercise* 2.—From a given point without a circle, whose distance from the circumference is not greater than the diameter, to draw a secant which shall be bisected by the circumference.**

PROP. XXXVII. THEOREM.

If from a point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it; and if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square of the straight line which meets it, that straight line touches the circle.

Let any point D be taken without the circle ABC, and from it let two straight lines DA and DB be drawn, of which DA cuts the circle and DB meets it. If the rectangle AD.DC be equal to the square of DB, DB touches the circle.



Draw the straight line DE, touching the circle ABC, in the point E (III. 17). Find F, the centre of the circle (III. 1); and join FE, FB, and FD.

Because FED is a right angle (III. 18), DE touches the circle ABC. But DA cuts the circle (*Hyp.*). Therefore the rectangle AD.DC is equal (III. 36) to the square of DE. But the rectangle AD.DC, is (*Hyp.*) equal to the square of DB. Therefore the square of DE is equal (I. *Ax.* 1) to the square of DB, and DE to DB. But FE is equal to FB (I. *Def.* 15). Wherefore the two sides DE and EF are equal to the two sides DB and BF, each to each; and the base FD is common to the two triangles DEF and DBF. Therefore the angle DEF is equal (I. 8) to the angle DBF. But DEF is a right angle. Therefore also DBF is a right angle (*Ax.* 1) and BF, if produced, is a diameter. But the straight line passing through the extremity of a diameter, at right angles to it (III. 16), touches the circle. Therefore DB touches the circle ABC. Wherefore, if from a point, &c. Q. E. D.

***Exercise.*—If tangents to a circle be drawn through the extremities of any two diameters which intersect each other, the straight line joining the intersections of these tangents will pass through the centre of the circle.**

The following Propositions may be added to this Book, as Exercises on different propositions contained in it. They will include necessary references also to the previous Books.

Prop. A. Theorem.—If any chord of a circle be produced, till the part produced be equal to the radius of the circle; and, if from the outward extremity of this secant, another secant be drawn through the centre of the circle; they will intercept arcs of the circumference, such that the convex arc is one-third of the concave arc.

Prop. B. Theorem.—If two straight lines intersect each other, within a circle, and cut the circumference, the angle at the point of their intersection is equal to half the angle at the centre standing on the sum of the opposite arcs intercepted between them; but, if they intersect each other without a circle, and either cut the circumference or touch it, the angle at the point of their intersection is equal to half the angle at the centre, standing on the difference of the arcs intercepted between them.

Prop. C. Theorem.—If the circumferences of a circle be cut by two straight lines, perpendicular to each other at any point, the squares of the four segments between that point and the points where they meet the circumference, are together equal to the square of the diameter.

Prop. D. Problem.—To divide a given straight line into two parts, such that the square of one of them may be equal to the rectangle contained by the other and a given straight line.

Prop. E. Problem.—To draw a straight line that shall touch two circles given in position, provided the one is not wholly within the other.

Prob. F. Theorem.—If the diameter of a given circle be produced, and two points be taken on opposite sides of the centre, such that the rectangle contained by their distances from the centre is equal to the square of the radius, any circle which passes through these points bisects the circumference of the given circle.

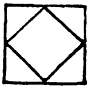
In concluding this Book, it may be remarked that Prop. XXXVI. suggests a mode of determining the diameter of the earth. For, in the figure to the first case of that proposition, if the circumference of the circle A B C represents that of the earth, A C the diameter of the earth, C D the altitude of any mountain above the level of the earth's surface, and D B the distance of the visible horizon; it is plain that if C D and D B be given in numbers, A C may be found, in numbers, from the nature of the proposition, by an easy arithmetical computation. The diameter of the earth being thus found to be nearly 8,000 miles (say exactly 7,920 miles), it may be proved by the application of this same proposition, that the distances of the visible horizon *in leagues*, are very nearly as the square roots of the altitudes *in fathoms;* that is, supposing the *altitudes* of the centre of the visible horizon to be 1, 4, 9, 16, 25, &c., *fathoms*, the *distances* of the visible horizon will be very nearly 1, 2, 3, 4, 5, &c. *leagues*. Hence, also conversely, the altitudes in fathoms are very nearly as the squares of the distances in leagues; that is, supposing the *distances* of the visible horizon to be 1, 2, 3, 4, 5, &c. *leagues*, the altitudes of its centre will be very nearly 1, 4, 9, 16, 25, &c. *fathoms*. This rule, of course, applies only to altitudes within the limits of the highest mountains on the surface of the earth.

BOOK IV.

DEFINITIONS.

I.

ONE rectilineal figure is said to be inscribed in another, when all the angular points of the inscribed figure are upon the sides of the figure in which it is inscribed, each upon each.



According to this definition, it is plain that the inscribed figure must have as many angles as the figure in which it is inscribed has sides, and consequently as many sides.

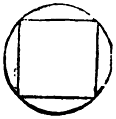
II.

In like manner, one rectilineal figure is said to be described about another, when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.

According to this definition also; it is plain that the circumscribed figure must have as many sides as the figure about which it is described has angles, and consequently as many angles.

III.

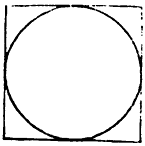
A rectilineal figure is said to be inscribed in a circle, when all the angular points of the inscribed figure are upon the circumference of the circle.



IV.

A rectilineal figure is said to be described about a circle, when each side of the circumscribed figure touches the circumference of the circle.

According to these definitions there is no limit to the number of the sides and angles of the rectilineal figure that may be inscribed in a circle, or described about it.

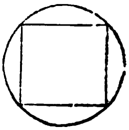


V.

In like manner, a circle is said to be inscribed in a rectilineal figure, when the circumference of the circle touches each side of the figure.

VI.

A circle is said to be described about a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure about which it is described.



VII.

A straight line is said to be placed in a circle, when the extremities of it are in the circumference of the circle.

The meaning of this definition is, that to place a straight line in a circle, is to draw a chord in the circle of a given length.

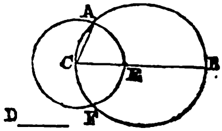
A rectilinear figure or polygon which has all its sides equal to one another is called *equilateral*; and that which has all its angles equal to one another is called *equiangular*.

A polygon which has all its sides and all its angles equal to one another, it called a *regular polygon*. Polygons receive particular names, according to the number of their sides and angles. Thus beginning with the triangle and the trapezium, for the sake of uniformity these are called the *trigon*, and the *tetragon*; but these terms are generally restricted to the equilateral triangle and the square. A polygon of *five* sides, is called a *pentagon*; of *six* sides, a *hexagon*; of *seven* sides, a *heptagon*; of *eight* sides, an *octagon*; &c. A polygon of *ten* sides is called a *decagon*; of twelve sides, a *duodecagon*; and of *fifteen* sides, a *quindecagon*, or more properly a *pentedecagon*.

PROP. I. PROBLEM.

In a given circle to place a straight line, equal to a given straight line which is not greater than the diameter.

Let ABC be the given circle, and D the given straight line, not greater than the diameter. It is required to place in the circle ABC a straight line equal to D.



Find the centre of the circle ABC (III. 1), and draw the diameter BC. If BC is equal to D, what is required is done; that is, in the circle ABC, a straight line BC is placed equal to D. But if BC is not equal to D, it is greater than D (*Hyp.*). From CB cut off CE equal to D (I. 3). From C as centre, at the distance CE, describe the circle AEF; and join CA.

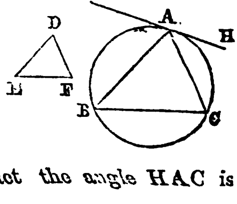
Because C is the centre of the circle AEF, CA is equal (I. *Def.* 15) to CE. But CE is equal (*Const.*) to D. Therefore CA is equal (I. *Ax.* 1) to D. Wherefore in the circle ABC, a straight line CA is placed equal to the given straight line D, which is not greater than the diameter of the circle. Q. E. F.

***Exercise.*—In a given circle, to place a straight line equal and parallel to a straight line given in position, and not greater than the diameter.**

PROP. II. PROBLEM.

In a given circle to inscribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle. It is required to inscribe in the circle ABC a triangle equiangular to the triangle DEF.



Draw the straight line GH touching the circle in the point A (III. 17). At the point A, in the straight line AH, make the angle HAC equal (I. 23) to the angle DEF. At the point A, in the straight line AG, make the angle GAB equal to the angle DFE. Join BC; the triangle ABC is the triangle required.

Because GH touches the circle ABC, and AC *is a chord* drawn from the point of contact the angle HAC is equal (III. 32) to the angle ABC in the alternate segment of the circle. But the angle HAC is equal to (*Const.*) the angle DEF. Therefore also the angle ABC is equal (I. *Ax.* 1) to DEF. For the same reason, the angle ACB is equal to the angle DFE. Therefore the remaining angle BAC is equal (I. 32 and *Ax.* 1) to the remaining angle EDF. Wherefore the triangle ABC is equiangular to the triangle DEF, and it is inscribed in the circle ABC. Q. E. F.

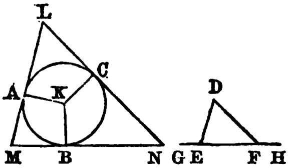
Exercise.—If a triangle be inscribed in one of two concentric circles, equiangular to a given triangle, it is required to inscribe the same in the other circle, so that its sides may be parallel to the sides of the former.

PROP. III. PROBLEM.

About a given circle to describe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle. It is required to describe about the circle ABC a triangle equiangular to the triangle DEF.

Produce BF both ways to the points G and H. Find the centre K of the circle ABC (III. 1), and from it draw any straight line KB. At the point K in the straight line KB, make the angle BKA equal (I. 23) to the angle DEG, and the angle BKC equal to the angle DFH. Through the points A, B, and C, draw (III. 17) the straight lines LM, MN, and NL, touching the circle ABC. The triangle LMN is the triangle required.



Because the straight lines LM, MN, and NL touch the circle ABC in the points A, B, and C, and the straight lines KA, KB, and KC, are drawn from the centre to the points of contact, the angles at the points A, B, and C (III. 18) are right angles. Because the four angles of the quadrilateral figure AMBK are equal to four right angles (I. 32, *Cor.* 8), and the two angles KAM and KBM are (*Const.*) right angles. Therefore the two angles AKB and AMB are equal (I. *Ax.* 3) to two right angles. But the angles DEG and DEF are equal (I. 13) to two right angles. Therefore the two angles AKB and AMB are equal (I. *Ax.* 1) to the two angles DEG and DEF. But the angle AKB is equal (*Const.*) to the angle DEG. Therefore the remaining angle AMB is equal (I. *Ax.* 3) to the remaining angle DEF. In like manner it can be shown that the angle LNM is equal to the angle DFE. Therefore the remaining angle MLN is equal (I. 32 and *Ax.* 3) to the remaining angle EDF. Wherefore the triangle LMN is equiangular to the triangle DEF; and it is described about the circle ABC. Q. E. F.

The construction and demonstration of this proposition would be as easily effected by first inscribing in the circle, a triangle equiangular to the given one, and then drawing tangents to the circle, through its three angular points. In the preceding demonstration, to prove that the tangents ML and NL must meet, it is only necessary to join AC, and apply the 9th and 12th Axioms of Book I. In the same way, it can be shown that MN and ML, as well as LN and MN must meet. The construction might be shortened by producing BK, a making the angles which the part produced makes with KA and KC, respe

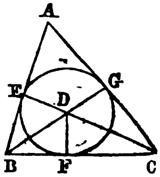
equal to the angles DEF and DFE in the triangle DEF, without producing EF. The construction and demonstration of the proposition in this way will form a useful exercise to the student.

PROP. IV. PROBLEM.

To inscribe a circle in a given triangle.

Let A B C be the given triangle. It is required to inscribe a circle in it.

Bisect the angles A B C and B C A by the straight lines B D and C D; and let these straight lines meet one another in the point D (I. 9). From the point D draw D E, D F, and D G perpendicular (I. 12) to A B, B C, and C A, respectively.



Because the angle E B D is equal (*Const.*) to the angle F B D, and the right angle B E D (I. *Ax.* 11) to the right angle B F D. Therefore the two triangles E B D and F B D have two angles of the one equal to two angles of the other, each to each; and the side B D, which is opposite to one of the equal angles in each, is common to both. Therefore their other sides are equal (I. 26), and D E is equal to D F. For the same reason, D G is equal to D F. Therefore D E is equal (I. *Ax.* 1) to D G, and the three straight lines D E, D F, and D G, are equal to one another. Wherefore the circle described from the centre D, at the distance of one of them, will pass through the extremities of the other two, and touch the straight lines A B, B C, and C A. Because the angles at the points E, F, and G, are right angles, and the straight line drawn through the extremity of a diameter at right angles to it, touches the circle (III. 16). Therefore the straight lines A B, B C, and C A touch the circle in the points E, F, and G. Wherefore the circle E F G is inscribed in the triangle A B C. Q. E. F.

The general problem which includes this proposition is:—To describe a circle touching three given straight lines which do not pass through the same point, and which are not all parallel to each other. If two of the straight lines be parallel, there may be two equal circles which fulfil the required conditions, viz., one on each side of the straight line which intersects the parallels. If the straight lines form a triangle, there will be four circles touching the straight lines, one inscribed as above, and the others touching the three sides externally and each of the other two sides produced. The examination and solution of these cases of this problem will form a very useful exercise.

***Corollary* 1.—The three straight lines which bisect the three angles of a triangle meet in the centre of the inscribed circle.**

***Corollary* 2.—In a right-angled triangle the diameter of its inscribed circle is equal to the difference between the sum of the legs and the hypotenuse; and the diameter of the circle which touches the hypotenuse, and the legs produced externally, is equal to the sum of the three sides, or the *perimeter* of the triangle.**

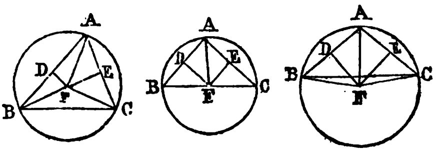
***Corollary* 3.—The area of any triangle is equal to the rectangle contained by the radius of the inscribed circle and half the sum of the three sides, or *semiperimeter* of the triangle.**

PROP. V. PROBLEM.

To describe a circle about a given triangle.

Let ABC be the given triangle. It is required to describe a circle ab

Bisect AB and AC in the points D and E (I. 10), and from these points draw DF and EF at right angles to AB and AC respectively (I. 11). The straight lines DF and EF produced meet one another. For, if DE were joined, the angles which DE would make with



them would be less than two right angles. Therefore (I. *Ax.* 12) they meet one another. Let them meet in F, and join FA. Also, if the point F be not in BC, join BF and CF.

Because AD, is equal to DB, and DF which is common to the two triangles ADF and BDF, is at right angles to AB, the base AF is equal (I. 4) to the base FB. In like manner, it may be shown that CF is equal to FA. Therefore BF is equal (I. *Ax.* 1) to FC; and FA, FB, and FC, are equal to one another. Wherefore the circle described from the centre F, at the distance of one of them, will pass through the extremities of the other two, and be described about the triangle ABC. Q. E. F.

COROLLARY.—It is manifest, that when the centre of the circle falls within the triangle, each of its angles is less than a right angle, each of them being in a segment greater than a semicircle; but, when the centre is one of the sides of the triangle, the angle opposite to this side, being in a semicircle, is a right angle; and, if the centre falls without the triangle, the angle opposite to the side beyond which it is, being in a segment less than a semicircle, is greater than a right angle (III. 31). Conversely, if the given triangle be acute-angled, the centre of the circle falls within it; if it be a right-angled triangle, the centre is in the side opposite to the right angle; and if it be an obtuse-angled triangle, the centre falls without the triangle, beyond the side opposite to the obtuse angle.

This proposition is virtually the same as that to describe a circle that shall pass through any three points not in the same straight line; or, as that to complete a circle of which an arc or a segment is given.

***Corollary.*—The three straight lines which bisect the sides of a triangle at right angles to them, meet in the centre of its circumscribed circle.**

The succeeding Proposition in this Book, may be greatly shortened by the following lemmas (things *taken* for granted, as proved) to which are added the demonstrations.

LEMMA 1.—Any regular polygon may have one circle described about it, and another inscribed in it: and the same point is the centre of both circles.

For, it is plain, first, that if all the angles of a regular polygon be bisected, the bisecting lines must all meet in the same point, and are all equal to one another; because it can be proved that any two straight lines which bisect two adjacent angles must meet (I. *Ax.* 12), and are equal to one another (I. 6). If therefore, a circle be described with this point as a centre at the distance of any one of these straight lines, it will pass through all the angular points of the *polygon*, and will be described about it (IV. Def. 2). It is plain, next, the

perpendiculars be drawn from the point where the bisecting lines meet, to all the sides of the polygon, these perpendiculars must be all equal to one another, because the sides of the polygon are equal chords of the circumscribed circle (III. 14). Therefore, if a circle be described with this point as a centre, at the distance of any one of these perpendiculars, it will pass through the extremities of all the others, and will touch the sides of the polygon (III. 16); because the angles at these extremities made by the sides of the polygon are right angles; and this circle will be inscribed in the polygon (IV. *Def.* 3).

Corollary.—The centre of the inscribed or circumscribed circle of any regular polygon, may be found by bisecting any two adjacent angles, or any two adjacent sides. For the point where the bisecting straight lines meet is the centre required.

LEMMA 2.—If any regular polygon be inscribed in a circle, tangents to the circle drawn through its angular points, will form a regular polygon of the same number of sides described about the circle.

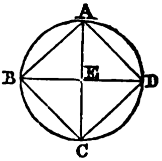
For, the angles which the tangents make with the sides of the inscribed figure, are equal (III. 32) to the angles in the alternate segments. But these segments are all equal (III. 28), because their chords are (*Hyp.*) equal. Therefore, all the triangles formed by the chords and the tangents are equal (I. 6 and 26) to one another; their sides are all equal, which are the halves of the sides of the circumscribed polygons; and all their third angles are equal, which are the angles of the circumscribed polygon. This polygon is therefore equilateral (I. 26) and equiangular, and it is described about the circle (IV. *Def.* 4).

PROP. VI. PROBLEM.

To inscribe a square in a given circle.

Let ABCD be the given circle. It is required to inscribe a square in it.

Draw the diameters, AC and BD, at right angles to one another (III. 1 and I. 11); and join AB, BC, CD, and DA. The quadrilateral figure ABCD is the square required.



Because the angles at the centre E are equal (*Const.*), the straight lines AB, BC, CD, and DA are all equal (III. 26). Therefore the quadrilateral figure ABCD is equilateral. Because the angles BAD, ABD, BCD, and CDA are all angles in a semicircle (*Const.*) they are all right angles (III. 31). Therefore the quadrilateral figure ABCD is rectangular. But it has been shown to be equilateral. Therefore it is a square (I. *Def.* 30), and it is inscribed in the circle ABCD. Q. E. F.

Corollary.—The square inscribed in a circle is double the square of its radius, or half the square of its diameter.

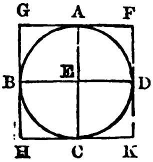
PROP. VII. PROBLEM.

To describe a square about a given circle.

Let ABCD be the given circle. It is required to describe a square about it.

Draw two diameters AC and BD of the circle ABCD, at right angles to one another (III. 1 and I. 11), and through the points A, B, C and D, draw the straight lines FG, GH, HK and KF, touching the circle (III. 17). The quadrilateral figure GHKF is the square required.

Because the angles at the points A, B, C, and D are right angles (*Const.*), G F is parallel to H K, and G H to F K (I. 28). Therefore G K, G D, and G C are parallelograms. But A C is equal to B D (*Const.*). Therefore, G F is equal to G H (I. *Ax.* 1), and the parallelogram G K is equilateral. Because G E is a parallelogram, and the angle A E B a right angle (*Const.*). Therefore, G E is a square (I. 46, *Cor.*), and the angle at G is a right angle. In the same manner, it can be shown that the angles at F, K, and H are right angles. Therefore the parallelogram G K is rectangular. But it was shown to be equilateral. Therefore G K is a square, and it is described about the circle A B C D. Q. E. F.



Otherwise.—Inscribe a square in the circle A B C D, and through its angular points draw tangents to the circumference. The figure formed by these tangents will be the square required (IV. *Lemma* 2).

Corollary 1.—The square described about the circle is the square of the diameter, and double of the inscribed square, or four times the square of the radius.

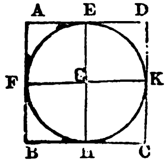
Corollary 2.—By bisecting the arcs of the circle subtending the sides of the inscribed square, and drawing chords to the different points of bisection from the angular points of the square, a regular *octagon* may be inscribed in the circle; and by drawing tangents through the angular points of the inscribed octagon, a regular octagon may be described about the circle. In the same manner, by the continuous bisection of arcs, a series of regular polygons may be inscribed in a circle, and described about it, the numbers of whose sides are successively 16, 32, 64, 128, &c.

PROP. VIII. PROBLEM.

To inscribe a circle in a given square.

Let A B C D be the given square. It is required to inscribe a circle in it.

Bisect each of the sides A B and A D in the points F and E (I. 10). Through E draw E H parallel to A B or D C (I. 31), and through F draw F K parallel to A D or B C. With centre G, and distance G E, describe the circle E F H K, and it is the circle required.



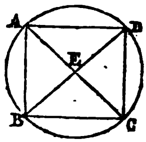
Because each of the figures A G, B G, G D, and G C is a square (I. 46 *Cor.* and *Ax.* 7), the four straight lines G E, G F, G H, and G K are equal to one another; and the circle described from the centre G at the distance of one of them G E, will pass through the extremities of the other three, and touch the straight lines A B, B C, C D, and D A. Because the angles at the points E, F, H, and K are right angles (I. 29), and the straight line drawn through the extremity of a diameter, at right angles to it, touches the circle (III. 16). Therefore each of the straight lines A B, B C, C D, and D A touches the circle. Wherefore the circle E F H K is inscribed in the square A B C D. Q. E. F.

Otherwise.—Find the centre of the inscribed circle (IV. *Lemma* 1, *Cor.*), and from it draw a perpendicular to one of the sides. With this centre and this perpendicular as radius describe a circle, and it will be the circle required.

PROP. IX. PROBLEM.

To describe a circle about a given square.

Let ABCD be the given square. It is required to describe a circle about it.



Join AC and BD, cutting one another in E. With centre E, and distance EA describe the circle ABCD, and it is the circle required.

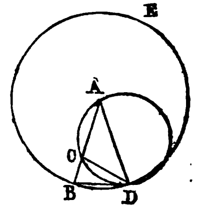
Because the diagonals of every parallelogram bisect each other, and the diagonals of a square are equal. Therefore the four straight lines EA, EB, EC, and ED are equal to one another (I. *Ax.* 7), and the circle described from the centre E, at the distance of one of them EA, will pass through the extremities of the other three. Wherefore the circle ABCD is described about the square ABCD. Q. E. F.

Otherwise.—Find the centre of the circumscribed circle (IV. *Lemma* 1. *Cor.*) and with this centre, and the distance of one of the angular points of the square as radius, describe a circle, and it will be the circle required.

PROP. X. PROBLEM.

To describe an isosceles triangle, having each of the angles at the base double of the third angle.

Take any straight line AB, and divide it in the point C (II. 11), so that the rectangle AB.BC may be equal to the square of CA. From the centre A, at the distance AB, describe the circle BDE. In this circle place the straight line BD equal to AC (IV. 1), and join DA. The triangle ABD has each of its angles ABD and ADB double of the angle BAD.



Join DC, and about the triangle ADC describe the circle ACD (IV. 5).

Because the rectangle AB.BC is equal to the square of AC, and AC is equal (*Const.*) to BD. Therefore, the rectangle AB.BC is equal (I. *Ax.* 1) to the square of BD. Because from the point B, without the circle ACD, two straight lines BA and BD are drawn to the circumference, one of which cuts and the other meets the circle, and the rectangle AB.BC, contained by the whole of the cutting line, and the part of it without the circle, is equal to the square of BD which meets it. Therefore the straight line BD touches the circle ACD (III. 37). Because BD touches the circle, and DC is drawn from the point of contact D, the angle BDC is equal (III. 32) to the angle DAC in the alternate segment of the circle. To each of these equals add the angle CDA. Therefore the whole angle BDA is equal (I. *Ax.* 2) to the two angles CDA and DAC. But the exterior angle BCD is equal (I. 32) to the two angles CDA and DAC. Therefore also BDA is equal (I. *Ax.* 1) to BCD. But BDA is equal (I. 5) to the angle CBD, because the side AD is equal to the side AB. Therefore the angle CBD, or DBA, is equal (I. *Ax.* 1) to the angle BCD. Wherefore the three angles BDA, DBA and BCD are equal to one another. Because the angle DBC is equal to the angle BCD, the side BD is equal (I. 6) to the side DC. But BD is equal to CA (*Const.*). Therefore also CA is equal (I. *Ax.* 1) to CD, and the angle CDA to the angle DAC (I. 5). Therefore the two angles CDA and DAC are together, double of the angle DAC. But BCD is equal (I. 32) to the two angles CDA and DAC. Therefore also the angle BCD is double of the angle DAC. But the angle BCD was proved to be equal to each of the angles BDA and DBA. Therefore each of the angles BDA and DBA is double of the angle DAB. Wherefore an isosceles triangle ABD has been described, having each of the angles at the base double of the third angle. Q. E. F.

The construction of this problem may be simplified by making a triangle ABD, having its three sides AB, AD, and BC, equal to three given straight lines, according to Prop. XXII. Book I.

Corollary 1.—The angle A at the vertex of the triangle ABD is one-fifth of two right angles, and each of the angles at the base two-fifths of two right angles.

Corollary 2.—The angle A is one-tenth of four right angles, the arc BD is one-tenth of the circumference, and the chord BD is the side of a regular decagon inscribed in the circle BDE.

Corollary 3.—A right angle may be *quinquisected*, that is, divided into five equal parts.

Corollary 4.—By doubling the arc BD one-fifth part of the circumference is obtained; and, by drawing its chord, the side of a regular pentagon inscribed in a circle, may be found.

Exercise 1.—To inscribe a regular decagon in a circle, and to describe another about it.

Exercise 2.—To inscribe a regular pentagon in a circle, and to describe another about it.

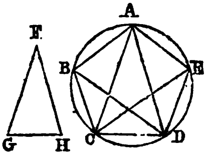
These exercises may be effected by the application of the preceding corollaries. Euclid's method is given in the next propositions.

PROP. XI. PROBLEM.

To inscribe a regular pentagon in a given circle.

Let ABCDE be the given circle. It is required to inscribe in it a regular pentagon.

Describe (IV. 10) an isosceles triangle FGH, having each of the angles at G and H double of the angle at F. In the circle ABCDE inscribe (IV. 2) the triangle ACD equiangular to the triangle FGH, so that the angle CAD may be equal to the angle at F, and the angles ACD and CDA equal to the angles at G and H. Therefore each of the angles ACD and CDA is double of the angle CAD. Bisect the angles ACD and CDA (I. 9) by the straight lines CE and DB. Join AB, BC, DE, and EA. The pentagon ABCDE is the pentagon required.



Because each of the angles ACD and CDA is double of the angle CAD, and they are bisected by the straight lines CE and DB, the five angles DAC, ACE, ECD, CDB, and BDA are equal to one another. But *equal angles* stand (III. 26) upon equal arcs. Therefore the five *arcs* AB, BC, CD, DE, and EA are equal to one another. But equal *arcs are subtended* (III. 29) by equal straight lines. Therefore the fi straight lines A B, B C, C D, D E, and E A are equal to one another. Wherefore the pentagon A B C D E is equilateral. Because the arc A B is equal to the arc D E. To each of these equals add the arc B C D. Therefore the whole arc A B C D is equal (I. *Ax.* 2) to the whole arc E D C B. But the angle A E D stands on the arc A B C D, and the angle B A E on the arc E D C B. Therefore the angle B A E is equal (III. 27) to the angle A E D. For the same reason, each of the angles A B C, B C D, and C D E is equal to the angle B A E, or A E D. Therefore the pentagon A B C D E is equiangular. But it has been shown to be equilateral. Therefore, in the given circle A B C D E, a regular pentagon A B C D E has been described. Q. E. F.

Corollary 1.—The interior angle of a pentagon is three-fifths of two right angles, or six-fifths of one right angle, that is, one right angle, and a fifth part of a right angle.

Corollary 2.—The exterior angle of a regular pentagon is two-fifths of two right angles or four-fifths of one right angle.

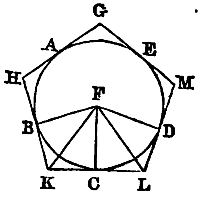
Exercise.—To describe a regular pentagon on a given straight line.

PROP. XII. PROBLEM.

To describe a regular pentagon about a given circle.

Let A B C D E be the given circle. It is required to describe a regular pentagon about it.

Find A, B, C, D, and E, the angular points of a regular pentagon inscribed in the circle (IV. 11). Through the points A, B, C, D, and E, draw the straight lines G H, H K, K L, L M, and M G touching the circle (III. 17). The pentagon G H K L M is the pentagon required. Take the centre F, and join F B, F K, F C, F L, and F D.



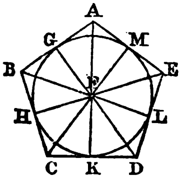
Because tangents drawn from the same point without a circle are equal, K B is equal to K C. Because in the two triangles K B F and K C F, the two sides B K and K F are equal to the sides C K and K F, and the base F B is equal to the base F C. Therefore the angle B K F is equal to the angle C K F (I. 8). In the same manner it may be shown that the angle C F L is equal to the angle D F L, and the angle C L F to the angle D L F. Because the angle B F C is equal to the angle C F D (III. 27). Therefore the angle K F C is equal to the angle L F C (I. *Ax.* 7). Because in the two triangles K F C and L F C, two angles K F C and K C F of the one are equal to two angles L F C and L C F of the other, and the side C F is common to both. Therefore the side C K is equal to the side C L, and the angle C K F is equal to the angle C L F (I. 26). Wherefore K L is double of K C, and the angle B K C double of F K C. In the same manner it may be shown that K H is double of K B, and the angle C L D double of C L F. Therefore K L is equal to K H, and the angle B K C to the angle C L D (I. *Ax.* 7). In the same manner it may be shown that H G, G M, and M L are each equal to H K or K L; and that the angles K H G, H G M, and G M L are each equal to the angle H K L or K L M. Therefore the pentagon G H K L M is both equilateral and equiangular. Wherefore a regular pentagon G H K L M has been described about the circle. Q. E. F.

Otherwise.—Find the angular points A, B, C, D, E, of a regular pentagon inscribed in the circle (IV. 11). Through these points draw tangents to the circle A B C D E, and they will form a regular pentagon described about the circle (IV. *Lemma* 2).

PROP. XIII. PROBLEM.

To inscribe a circle in a given regular pentagon.

Let A B C D E be the given regular pentagon. It is required to inscribe a circle in it.



Bisect the angles B C D and C D E by the straight lines C F and D F (I. 9). From the point F, in which they meet, draw the straight lines F B, F A, and F E. Draw also the perpendiculars F G, F H, F K, F L, and F M to the sides of the pentagon. From centre F, with distance F H, describe the circle GHKLM, and it is inscribed in the pentagon.

Because the angles F C D and F D C (I. *Ax.* 7) are equal, F C is equal to F D. Because in the two triangles B C F and D C F, the side B C is equal to the side D C (*Hyp.*), C F is common, and the angle B C F is equal to the angle D C F (*Const.*). Therefore B F is equal to D F, and also to C F. The angle C B F is also equal to the angle C D F, and is the half of the angle A B C. In the same manner it may be shown that the angles A and E are bisected by the straight lines A F and E F. Because the two angles F C H and F C K are equal (*Const.*), and the two angles F H C and F K C are also equal, being right angles (*Const.*). Therefore in the two triangles F H C, F K C, two angles of the one are equal to two angles of the other, and the side F C is common to both. Wherefore, the two triangles are equal (I. 26), and F H is equal to F K. In the same manner, it may be shown that F L, F M, and F G, are each of them equal to F H, or F K. Therefore the five straight lines F G, F H, F K, F L, and F M are equal to one another, and the circle described from the centre F, at the distance of one of them F H, will pass through the extremities of the other four, and touch the straight lines A B, B C, C D, D E, and E A. Because the angles at the points G, H, K, L, and M are right angles, and a straight line drawn through the extremity of the diameter of a circle at right angles to it (III. 16) touches the circle. Therefore each of the straight lines A B, B C, C D, D E, and E A touches the circle. Wherefore the circle G H K L M is inscribed in the pentagon A B C D E. Q. E. F.

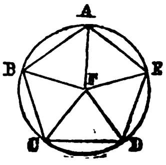
Otherwise.—Find F the centre of the inscribed circle (IV. *Lemma* 1, *Cor.*), and with radius F H describe the circle G H K L M, and it is inscribed in the pentagon A B C D E.

PROP. XIV. PROBLEM.

To describe a circle about a given regular pentagon.

Let A B C D E be the given regular pentagon. It is required to *describe a circle* about it.

Bisect the angles B C D and C D E by the straight lines C F and F D (I. 9). From the point F, in which they meet, draw the straight lines F B, F A, and F E. With centre F, and distance F C, describe the circle A B C D E, and it is described about the pentagon.



Because it may be shown, as in the preceding proposition, that the side C F is equal to the side F D, and that F B, F A, and F E are each of them equal to F C or F D. Therefore the five straight lines F A, F B, F C, F D, and F E are equal to one another. And the circle described from the centre F, at the distance of one of them, will pass through the extremities of the other four. Wherefore the circle A B C D E is described about the pentagon A B C D E. Q. E. F.

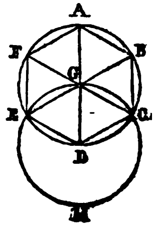
Otherwise.—Find F the centre of the circumscribed circle (IV. *Lemma* 1, *Cor.*), and with radius F A describe the circle A B C D E, and it is described about the pentagon A B C D E.

PROP. XV. PROBLEM.

To inscribe a regular hexagon in a given circle.

Let A B C D E F be the given circle. It is required to inscribe a regular hexagon in it.

Find the centre G of the circle A B C D E F (III. 1), and draw the diameter A G D. From D, as a centre, at the distance D G, describe the circle E G C H, join E G and C G, and produce them to the points B and F. Join A B. B C, C D, D E, E F, and F A. The hexagon A B C D E F is a regular hexagon.



Because G is the centre of the circle A B C D E F, G E is equal to G D. Because D is the centre of the circle E G C H, D E is equal to D G. Therefore G E is equal to E D (I. *Ax.* 1), and the triangle E G D is equilateral. Because the three angles E G D, G D E, and D E G, are equal to one another (I. 5, *Cor.*). But the three angles of a triangle are equal to two right angles (I. 32). Therefore the angle E G D is the third part of two right angles. In the same manner it may be shown that the angle D G C is also the third part of two right angles. Because the straight line G C makes with E B the adjacent angles E G C, C G B equal to two right angles (I. 13), the remaining angle C G B is the third part of two right angles. Therefore the angles E G D, D G C, and C G B are equal to one another. And the vertical angles B G A, A G F, and F G E (I. 15) are equal to these angles, each to each. Therefore the six angles E G D, D G C, C G B, B G A, A G F, and F G E are equal to one another. But equal angles stand upon equal arcs (III. 26). Therefore the six arcs A B, B C, C D, D E, E F, and F A are equal to one another. And equal arcs are subtended by equal straight lines (III. 29). Therefore the six straight lines A B, B C, C D, D E, E F, and F A are equal to one another, and the hexagon A B C D E F is equilateral. Again, because the arc A F is equal to the arc E D. To each of these equals, add the arc A B C D. Therefore the whole arc F A B C D is equal to the whole arc E D C B A. But the angle F E D stands upon the arc F A B C D, and the angle

A F E upon the arc E D C B A. Therefore the angle A F E is equal to the angle F E D (III. 27). In the same manner it may be shown that the other angles of the hexagon A B C D E F are each equal to the angle A F E or F E D. Therefore the hexagon is equiangular. And it was shown to be equilateral. Therefore the regular hexagon A B C D E F is inscribed in the given circle A B C D E F. Q. E. F.

COR. 1.—From this it is manifest, that the side of the hexagon is equal to the straight line drawn from the centre to the circumference, that is, to the radius or semidiameter of the circle.

If through the points A, B, C, D, E, F there be drawn straight lines touching the circle, a regular hexagon will be described about it: and a circle may be inscribed in a given regular hexagon, and circumscribed about it in the same manner as was done in the case of the regular pentagon.

Otherwise.—Find the centre G of the circle A B C D E F, and draw any radius A G. Draw the chord A B equal to A G (IV. 1), and join B G. Because the triangle A G B is equilateral, the angle A G B is one-third of two right angles (I. 32), or one-sixth of four right angles. But all the angles at G are equal to four right angles. Therefore, the arc A B is one-sixth of the circumference. Draw chords equal to A B, in the circumference all round the circle, and contiguous to each other (IV. 1),—viz., BC, CD, DE, EF, and F A. The figure ABCDEF is a regular hexagon.

Because the chords AB, B C, C D, D E, E F, and F A are all equal, the hexagon is equilateral. Because each of its angles stands upon four-sixths of the circumference, they are equal to each other (III. 27). Therefore the hexagon is equiangular. Wherefore in the circle A B C D E F a regular hexagon is inscribed.

Corollary 2.—The interior angles of a regular hexagon are each equal to two-thirds of two right angles, or to a right angle and a third of a right angle.

Corollary 3.—A regular hexagon may be described on a given straight line.

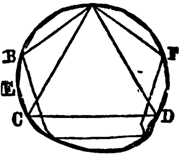
Corollary 4.—The area of the regular hexagon is six times that of the equilateral triangle described on the same straight line.

Corollary 5.—If the alternate angular points of a regular hexagon be joined, it will form an equilateral triangle. Thus, if A, E, and C be joined an equilateral triangle will be inscribed in the circle; and if tangents to the circle be drawn through these points, an equilateral triangle will be described about it.

PROP. XVI. PROBLEM.

To inscribe a regular quindecagon in a given circle.

Let A B C D be the given circle. It is required to inscribe a regular quindecagon in it.



Find A C the side of an equilateral triangle inscribed in the circle (IV. 2), and A B the side of a regular pentagon inscribed in the same (IV. 11). Bisect B C in E (III. 30). Join B E and E C, and place (IV. 1) in the circumference straight lines equal to these, and contiguous to each other, all round the circle. The figure A B C D F is a regular quindecagon.

Because of such equal parts as the whole circumference A B C D F contains fifteen, the arc A B C, which is the third part of the whole, contains five; *and the arc A B,* which is the fifth part of the *whole, contains* three. Therefore their difference B C contains two the same parts, and B E, E C are, each of them, the fifteenth part of the whole circumference A B C D, and the figure A B E C D F is equilateral. Because each of its angles stands upon thirteen-fifteenths of the circumference, it is also equiangular (III. 27). Therefore a regular quindecagon is inscribed in the circle A B C. Q. E. F.

In the same manner as was done in the pentagon, if through the angular points of the inscribed quindecagon, straight lines be drawn touching the circle, a regular quindecagon will be described about it; and likewise, as in the case of the pentagon, a circle may be inscribed in a given regular quindecagon, and circumscribed about it.

Otherwise.—Find two sides of the regular pentagon, and one side of the regular trigon inscribed in the circle, and commencing at the same point in the circumference. Join the points in which the second side of the pentagon and the side of the trigon terminate. The chord thus drawn will be the side of the inscribed pentedecagon or quindecagon. Because of such equal parts as the whole circumference contains fifteen, the arc subtended by the two sides of the regular pentagon contains two-fifths or six, and the arc subtended by the side of the trigon contains one-third or five. Therefore their difference contains one-fifteenth or one of those parts, which is the side of the quindecagon required.

BOOK V.

DEFINITIONS.

I.

A LESS magnitude is said to be *a part* of a greater magnitude, when the less measures the greater; that is, when the less is contained a certain number of times exactly in the greater.

The term *part* is here evidently understood in a restricted sense, and one which is commonly expressed by the phrase *aliquot part*. Better terms are *measure* or *submultiple*, either of which signifies the same as *part* or *aliquot part*, and conveys a more distinct notion of the meaning.

II.

A greater magnitude is said to be a multiple of a less, when the greater is measured by the less; that is, when the greater contains the less a certain number of times exactly.

The meaning of this definition, taken in connexion with the preceding one, will be best understood by adopting two distinct terms which are correlative; viz., *multiple*, and *submultiple*. Thus, if one magnitude contains another an *exact number* of times, the greater magnitude is called the *multiple* of the smaller; and the smaller, the *submultiple* of the greater.

III.

The mutual relation of two magnitudes of the same kind to one another, in respect of quantity, is called their ratio.

The term *ratio* is employed to express the relation of two like magnitudes to each other, whether they be *commensurable* or *incommensurable*, that is, whether they have a common *measure* or not. Thus, the diagonal of a square has a certain ratio to the side of the square; but this ratio cannot be expressed, like many others, in commensurate terms; for their common measure, or common unit is unknown.

IV

Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other.

This definition is intended as a test of the likeness or similarity of any two magnitudes; for unless the one can be multiplied so as to exceed the other in magnitude, they cannot be said to be of the *same kind*, and so cannot have any ratio to each other.

V.

The first of four magnitudes is said to have the same ratio to the second, which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth; if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the *fourth: or,* if the multiple of the first be equal to that of the *second, the* multiple of the third is also equal to that of the fourth: or if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.

This is the most important definition of the Fifth Book. Upon it, as a hinge or centre, turns the whole doctrine of proportion delivered in this Book, and applied in the sixth and subsequent Books. Volumes have been written to explain its meaning, and yet after all it is very simple. It is plain in the first place, that of any four magnitudes such as are spoken of in the definition, the first two must be *homogeneous*, or both of the same kind, and the last two must be *homogeneous*, or both of the same kind; but these two kinds may be different in themselves; that is, each pair may be *heterogeneous*, or of a different kind. Secondly, it is plain that by *equimultiples* of two magnitudes, is meant that each magnitude is taken or repeated the *same number of times*. Now, by taking equimultiples of the first and third of the supposed magnitudes, and equimultiples of the second and fourth of the same magnitudes, if it can be shown from the nature of the case to which this test is applied, that when the multiple of the first is greater than that of the second, the multiple of the third must also be greater than that of the fourth; or that when the multiple of the first is equal to that of the second, the multiple of the third must be equal to that of the fourth; or, lastly, when the multiple of the first is less than that of the second, the multiple of the third must be less than that of the fourth; then, it necessarily follows according to this definition, that the first has to the second the same ratio that the third has to the fourth; that is, that there is an equality of ratios between the first pair and the second pair of magnitudes. The application of this definition to particular cases, however, will be sure to render it much more clear and distinct to the learner.

VI.

Magnitudes which have the same ratio are called proportionals. N.B.—"When four magnitudes are proportionals, this property is usually expressed by saying, the first is to the second, as the third to the fourth."

This definition merely explains the term *proportional* as applied to magnitudes such as are supposed in the preceding definition, which constitutes the *test of proportionality*.

VII.

When of the equimultiples of four magnitudes (taken as in the fifth definition), the multiple of the first is greater than that of the second, but the multiple of the third is not greater than the multiple of the fourth; then the first is said to have to the second a greater ratio than the third magnitude has to the fourth: and, on the contrary, the third is said to have to the fourth a less ratio than the first has to the second.

This definition becomes plain and easy after the fifth definition is understood. It may be be considered as a *test of non-proportionality*,

VIII.

Analogy, or proportion, is the similitude of ratios.

This definition has been greatly objected to. Much of the objection may be removed by adopting the word *sameness* or *equality*, instead of *similitude*, this change being justified by the phraseology of the fifth definition itself.

IX.

Proportion consists in three terms at least.

When a proportion consists of three terms their order is continual, or such that the *first has the same* ratio to the second which the second has to the third.

X.

When three magnitudes are continual proportionals, the first is said to have to the third, the duplicate ratio of that which it has to the second.

When three magnitudes are continual proportionals, the ratio of the first to the third is compounded of *two equal* ratios—viz., the ratio of the first to the second, and the ratio of the second to the third; hence, it is called *duplicate* ratio.

XI.

When four magnitudes are continual proportionals, the first is said to have to the fourth, the triplicate ratio of that which it has to the second, and so on; quadruplicate, &c., increasing the denomination still by unity in any number of proportionals.

When four magnitudes are continual proportionals, the ratio of the first to the fourth is compounded of *three equal* ratios,—viz., the ratio of the first to the second, the ratio of the second to the third, and the ratio of the third to the fourth; hence, it is called *triplicate* ratio. In like manner, *quadruplicate* ratio is a ratio compounded of *four equal* ratios, &c.

A.

When there are any number of magnitudes of the same kind, the first is said to have to the last of them the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on unto the last magnitude.

For example, if A, B, C, D be four magnitudes of the same kind, the first A is said to have to the last D, the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D; or, the ratio of A to D is said to be compounded of the ratios of A to B, B to C, and C to D.

And if A has to B the same ratio which E has to F; and B to C the same ratio that G has to H; and C to D the same that K has to L; then, by this definition, A is said to have to D the ratio compounded of ratios which are the same with the ratios of E to F, G to H, and K to L. And the same thing is to be understood when it is more briefly expressed by saying, A has to D the ratio compounded of the ratios of E to F, G to H, and K to L.

In like manner, the same things being supposed, if M has to N the same ratio which A has to D; then, for shortness' sake, M is said to have to N the ratio compounded of the ratios of B to F, G to H, and K to L.

This definition marked A, is usually called *the definition of compound ratio.* It was supplied by Dr. Simson, and was considered by him to have originally belonged to the Elements, though not in the Greek text.

XII.

In proportionals, the antecedent terms are called homologous to one another, as also the consequents to one another.

Proportionals consist of a series of ratios. In any ratio, which, of course, consists of two *terms* or *magnitudes*, the *first* term of the ratio is called the *antecedent*, and the *second* term the *consequent*. In an ordinary proportion consisting of *four* terms, the *first* and the *third*, being the *antecedents* of the two ratios, are called *homologous* terms,—that is, terms which *agree with one another* as to their name; and the *second* and the *fourth* being the consequents of the two ratios, are also called *homologous* terms.

"*Geometers* make use of the following technical words or phrases to signify certain ways of changing either the order or magnitude of proportionals, so that they continue still to be proportionals."

[The memory need not be burdened with these explanations until the propositions be studied to which they refer.]

XIII.

Permutando, or *alternando*, by permutation or alternately. This phrase is used when there are four proportionals, and it is inferred that the first has the same ratio to the third which the second has to the fourth; or that the first is to the third as the second to the fourth: as is shown in Prop XVI. of this Fifth Book.

XIV.

Invertendo, by inversion; when there are four proportionals, and it is inferred that the second is to the first as the fourth to the third.—Prop. B. Book V.

XV.

Componendo, by composition; when there are four proportionals, and it is inferred that the first together with the second, is to the second, as the third together with the fourth, is to the fourth.—Prop. 18, Book V.

XVI.

Dividendo, by division; when there are four proportionals, and it is inferred that the excess of the first above the second, is to the second, as the excess of the third above the fourth, is to the fourth.—Prop 17, Book V.

XVII.

Convertendo, by conversion; when there are four proportionals, and it is inferred, that the first is to its excess above the second, as the third to its excess above the fourth.—Prop E. Book V.

XVIII.

Ex æquali (sc. distantiâ), or *ex æquo*, from equality of distance: when there is any number of magnitudes more than two, and as many others, such that they are proportionals when taken two and two of each rank, and it is inferred that the first is to the last of the first rank of magnitudes, as the first is to the last of the others: "Of this there are the two following kinds, which arise from the different order in which the magnitudes are taken, two and two."

XIX.

Ex æquali, from equality. This term is used simply by itself, when the first magnitude is to the second of the first rank, as the first to the second of the other rank; and as the second is to the third of the first rank, so is the second to the third of the other; and so on in order: and the inference is as mentioned in the preceding definition; whence this is called ordinate proportion. It is demonstrated in Prop. 22, Book V.

XX.

Ex æquali in proportione perturbatâ seu inordinatâ, from equality in perturbate or disorderly proportion (*Prop. 4, Lib. II. Archimedis de sphærâ et cylindro*). The term is used when the first magnitude is to the second of the first rank, as the last but one is to the last of the second rank; and as the second is to the third of the first rank, so is the last but two to the last but one of the second rank; and as the third is to the fourth of the first rank, so is the third from the last to the last but two of the second rank; and so on in a cross order: and the inference is as in the 18th definition. It is demonstrated in Prop. 23, Book V.

[The Latin terms explained in these eight definitions are usually replaced by the English words which express their meaning. But they are retained in most editions of Euclid, as being useful in reading good old authors on geometry.]

AXIOMS.

I.

Equimultiples of the same, or of equal magnitudes, are equal to one another.

II.

Those magnitudes, of which the same or equal magnitudes are equimultiples, are equal to one another.

This axiom means that equi-submultiples of the same or of equal magnitudes, are equal.

III.

A multiple of a greater magnitude is greater than the same multiple of a less.

IV.

Of two magnitudes, that one of which a multiple is greater than the same multiple of the other, is the greater.

V.

A part or submultiple of a greater magnitude is greater than the same part or submultiple of a less magnitude.

VI.

Of two magnitudes, that one of which a part or submultiple is greater than the same part or submultiple of the other, is the greater of the two.

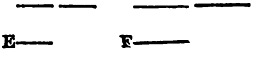
These two axioms are not Euclid's, but they are added as useful for reference.

The magnitudes in the Fifth Book are usually represented by straight lines, for the sake of simplicity; but any other kind of figures may be employed to indicate magnitudes in general.

PROP. I. THEOREM.

If any number of magnitudes be equimultiples of as many other magnitudes, each of each; what multiple soever any one of the first magnitudes is of its part, the same multiple is all the first magnitudes of all the other magnitudes.

Let any number of magnitudes AB and CD be equimultiples of as many others E and F, each of each. Whatsoever multiple AB is of E, the same multiple is AB and CD together, of E and F together.



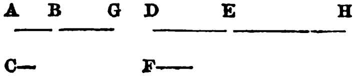
Divide AB into magnitudes each equal to E, viz. AG and GB; and CD into CH and HD, each equal to F. Because the number of the magnitudes CH and HD is equal to the number of the others AG and G B (*Hyp.*); and A G is equal to E, and C H to F (*Const.*). Therefore A G and C H together are equal (I. *Ax.* 2) to E and F together. Because G B is equal to E, and H D to F. Therefore G B and H D together are equal to E and F together. Wherefore as many magnitudes as A B contains each equal to E, so many do A B and C D together contain each equal to E and F together. Therefore, whatsoever multiple A B is of E, the same multiple is A B and C D together, of E and F together. The same demonstration holds in any number of magnitudes, which is here applied to two. Therefore, if any number of magnitudes, be equimultiples of as many others, each of each; whatsoever multiple any one of them is of its part, the same multiple is all the first magnitudes of all the others. Q. E. D.

PROP. II. THEOREM.

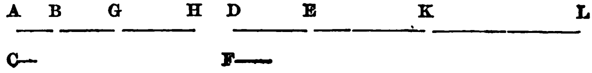
If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth; then the first together with the fifth is the same multiple of the second, that the third together with the sixth is of the fourth.

Let A B the first, be the same multiple of C the second, that DE the third, is of F the fourth; and BG the fifth, the same multiple of C the second, that E H the sixth, is of F the fourth. A G, the first together with the fifth, is the same multiple of C the second, that D H, the third together with the sixth, is of F the fourth.

Because A B is the same multiple of C that D E is of F (*Hyp.*), A B contains as many magnitudes each equal to C, as D E contains each equal to F. For the same reason, B G contains as many each equal to C, as E H contains each equal to F. Therefore the whole A G contains as many each equal to C, as the whole D H contains each equal to F. Therefore A G is the same multiple of C that D H is of F; that is, A G, the first and fifth together, is the same multiple of the second C, that D H, the third and sixth together, is of the fourth F. If, therefore, the first be the same multiple, &c. Q. E. D.



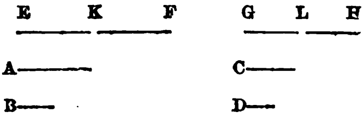
COROLLARY.—From this it is plain, that if any number of magnitudes A B, B G, G H be multiples of another C; and as many D E, E K, K L be the same multiples of F, each of each; the whole of the first,—viz., A H, is the same multiple of C, that the whole of the last,—viz., D L, is of F.



PROP. III. THEOREM.

If the first be the same multiple of the second, which the third is of the fourth; and if of the first and third there be taken equimultiples; these are equimultiples, the one of the second, and the other of the fourth.

Let A *the first,* be the same multiple of B the second, that C the third, is of D the fourth; and of A and C let equimultiples E F and G H be taken. E F is the same multiple of B, that G H is of D.



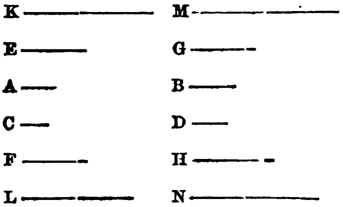
Divide EF into the magnitudes E K and K F, each equal to A; and G H into G L and L H, each equal to C.

Because the number of the magnitudes E K and K F is equal to the number of the others G L and L H; and A is the same multiple of B, that C is of D. But E K is equal to A, and G L equal to C. Therefore E K is the same multiple of B, that G L is of D. For the same reason, K F is the same multiple of B, that L H is of D. Because the first E K is the same multiple of the second B, which the third G L is of the fourth D, and the fifth K F is the same multiple of the second B, which the sixth L H is of the fourth D. Therefore, E F the first together with the fifth, is the same multiple of the second B (V. 2), which G H the third together with the sixth, is of the fourth D. The same demonstration holds, if there be more parts in E F and G H, each equal to A and C respectively. If, therefore, the first, &c. Q. E. D.

PROP. IV. THEOREM.

If the first of four magnitudes has the same ratio to the second which the third has to the fourth; any equimultiples whatever of the first and third have the same ratio to any equimultiples of the second and fourth; viz., "the equimultiple of the first has the same ratio to that of the second, which the equimultiple of the third has to that of the fourth."

Let A the first have to B the second, the same ratio which the third C has to the fourth D; and of A and C let there be taken any equimultiples whatever E and F; and of B and D any equimultiples whatever G and H. E has the same ratio to G, which F has to H.



Take of E and F any equimultiples whatever K and L, and of G and H any equimultiples whatever M and N.

Because E is the same multiple of A, that F is of C; and of E and F equimultiples K and L have been taken. Therefore K is the same multiple of A, that L is of C (V. 3). For the same reason, M is the same multiple of B, that N is of D. Because, as A is to B, so is C to D (*Hyp.*), and of A and C certain equimultiples K and L have been taken, and of B and D certain equimultiples M and N have been taken. Therefore if K be greater than M, L is greater than N; if equal, equal; and if less, less (V. *Def.* 5). But K and L are any equimultiples whatever of E and F (*Const.*), and M and N any whatever of G and H. Therefore as E is to G, so is F to H (V. *Def.* 5). Therefore, if the first, &c. Q. E. D.

Corollary.—Likewise, if the first has the same ratio to the second, *which the third* has to the fourth, then also any equimultiples whatever of the first and third shall have the same ratio to the second and fourth; and in like manner, the first and the third shall have the same ratio to any equimultiples whatever of the second and fourth.

Let A the first, have to B the second, the same ratio which the third B has to the fourth D, and of A and C let E and F be any equimultiples whatever. Then E shall be to B as F to D.

Take of E and F any equimultiples whatever, K and L; and of B and D any equimultiples whatever, G and H.

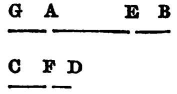
It may be shown as before, that K is the same multiple of A, that L is of C. Because A is to B, as C is to D (*Hyp.*), and of A and C certain equimultiples have been taken,—viz., K and L; and of B and D certain equimultiples G and H. Therefore, if K be greater than G, L is greater than H; if equal, equal; and if less, less (V. *Def.* 5). But K and L are any equimultiples whatever of E and F (*Const.*), and G and H any whatever of B and D. Therefore, E is to B, as F is to D (V. *Def.* 5). And in the same way the other case is demonstrated.

PROP. V. THEOREM.

If one magnitude be the same multiple of another, which a part first taken from the first is of a part taken from the other; the first remainder is the same multiple of the second remainder, that the one magnitude is of the other.

Let the magnitude AB be the same multiple of the magnitude CD, that AE taken from the first, is of CF taken from the other. The remainder EB is the same multiple of the remainder FD, that AB is of CD.

Take AG the same multiple of FD, that AE is of CF.

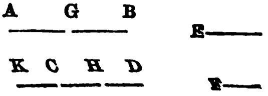


Because AE is the same multiple of CF, that EG is of CD (V. 1). But AE (*Hyp.*) is the same multiple of CF, that AB is of CD. Therefore EG is the same multiple of CD, that AB is of CD; and EG is equal to AB (V. *Ax.* 1). Take from each of these equals the common magnitude AE; and the remainder AG is equal, to the remainder EB. But AE is the same multiple of CF, that AG is of FD (*Const.*), and AG is equal to EB. Therefore AE is of the same multiple of CF, that EB is of FD. But AE is the same multiple of CF, that AB is of CD (*Hyp*). Therefore EB is the same multiple of FD, that AB is of CD. Therefore, if one magnitude, &c. Q. E. D.

PROP. VI. THEOREM.

If two magnitudes be equimultiples of two other magnitudes, and if equimultiples of the second magnitudes be taken from the first; the remainders are either equal to the second magnitudes, or equimultiples of them.

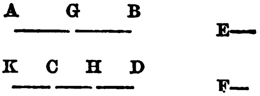
Let the two magnitudes AB and CD be equimultiples of the two magnitudes E and F; and let AG and CH, taken from the first, be also equimultiples of E and F. The remainders GB and HD are either equal to E and F, *or equimultiples of them.*



First, let GB be equal to E. HD is equal *to F. Make CK* equal to F.

Because A G is the same multiple of E, that CH is of F (*Hyp.*); and G B is equal to E, and C K to F. Therefore A B is the same multiple of E, that KH is of F. But A B (*Hyp.*) is the same multiple of E, that C D is of F. Therefore K H is the same multiple of F, that CD is of F; and K H is equal to C D (V. *Ax.* 1). From these equals, take the common magnitude C H. The remainder K C is equal to the remainder H D. But K C is equal to F (*Const.*). Therefore H D is equal to F.

Next let G B be a multiple of E. H D is the same multiple of F. Make C K the same multiple of F, that G B is of E.



Because A G is the same multiple of E, that C H is of F (*Hyp.*); and G B the same multiple of E, that C K is of F. Therefore A B is the same multiple of E, that K H is of F (V. 2). But A B is the same multiple of E, that C D is of F; (*Hyp.*). Therefore K H is the same multiple of F, that C D is of F; and K H is equal to CD (V. *Ax.* 1). From these equals, take C H, and the remainder K C is equal to the remainder H D. Because G B is the same multiple of E, that K C is of F (*Const.*), and K C is equal to HD. Therefore H D is the same multiple of F, that GB is of E. If, therefore, two magnitudes, &c. Q. E. D.

The four following propositions marked A, B, C, D, were introduced by Dr. Simson, as properly belonging to this Book, though not found in the original Greek; and, as necessary to complete the demonstration of some subsequent propositions.

PROP. A. THEOREM.

If the first of four magnitudes has the same ratio to the second, which the third has to the fourth; and if the first be greater than the second, the third is also greater than the fourth; if equal, equal; and if less, less.

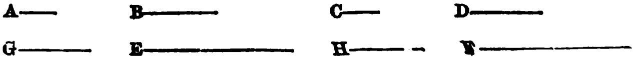
Take any equimultiples of each of the magnitudes; as the doubles of each.

Because (*Def.* 5.), if the double of the first be greater than the double of the second, the double of the third is greater than the double of the fourth. But, if the first be greater than the second, the double of the first is greater than the double of the second, and the double of the third is greater than the double of the fourth. Therefore the third is greater than the fourth. In like manner, if the first be equal to the second, or less than the second, the third can be proved equal to the fourth, or less than the fourth. Therefore, if the first, &c. Q. E. D.

PROP. B. THEOREM.

If four magnitudes be proportionals, they are proportionals when taken inversely.

Let A be to B, as C is to D. Inversely (*Def.* 14), B is to A as D is to C.



Take of B and D any equimultiples whatever K and F; and of A and *C any equimultiples* whatever G and H.

First, let E be greater than G, then G is less than E. Because A is to B, as C is to D (*Hyp.*), and of A and C, the first and third, G and H are equimultiples; and of B and D, the second and fourth, E and F are equimultiples. But G is less than E. Therefore H is less than F (V. *Def.* 5), that is F is greater than H. Wherefore, if E be greater than G, F is greater than H. In like manner, if E be equal to G, F may be shown to be equal to H; and if less, less. But E and F, are any equimultiples whatever of B and D (*Const.*); and G and H any whatever of A and C. Therefore, as B is to A, so is D to C (V. *Def.* 5). Therefore, if four magnitudes, &c. Q. E. D.

PROP. C. THEOREM.

If the first be the same multiple of the second, or the same part of it, that the third is to the fourth; the first is to the second, as the third is to the fourth.

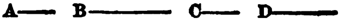
Let the first A, be the same multiple of the second B, that the third C, is of the fourth D. Then A is to B as C is to D.

Take of A and C any equimultiples whatever E and F; and of B and D any equimultiples whatever G and H.

Because A is the same multiple of B that C is of D (*Hyp.*), and E the same multiple of A, that F is of C (*Const.*). Therefore E is the same multiple of B, that F is of D (V. 3), that is, E and F are equimultiples of B and D. But G and H are equimultiples of B and D (*Const.*). Therefore if E be a greater multiple of B than G is of B, F is a greater multiple of D than H is of D; that is, if E be greater than G, F is greater than H. In like manner, if E be equal to G, or less than G, it may be shown that F is equal to H, or less than H. But E and F are any equimultiples whatever of A and C (*Const.*); and G and H any equimultiples whatever of B and D. Therefore A is to B, as C is to D (V. *Def.* 5).

Next, let the first A be the same part of the second B, that the third C is of the fourth D. A is to B, as C is to D.

Because A is the same part of B that C is of D. Therefore B is the same multiple of A that D is of C. Wherefore, by the preceding case, B is to A, as D is to C. Therefore inversely, A is to B, as C is to D (V. B.). Wherefore, if the first be the same multiple, &c. Q. E. D.



PROP. D. THEOREM.

If the first be to the second as the third is to the fourth, and if the first be a multiple, or a part of the second; the third is the same multiple, or the same part of the fourth.

Let A be to B as *C* is to D.

In the first place, let A the first be a multiple of B the second. The third C is the same multiple of the fourth D.

Take E equal to A, and whatever multiple A or E is of B, make F the same multiple of D.

Because A is to B, as C is to D (*Hyp.*); and of B the second, and D the fourth, equimultiples have been taken, E and F. Therefore A is to E, as C is to F (V. 4, *Cor.*). But A is equal to E (*Const.*). Therefore C is equal to F (V. A). But F is the same multiple of D that A is of B (*Const.*). Therefore C is the same multiple of D that A is of B.

Next, let A the first, be a part of B the second. The third C is the same part of the fourth D.

Because A is to B, as C is to D (*Hyp.*). Inversely, B is to A, as D to C. (V. B). But A is a part of B (*Hyp.*). Therefore B is a multiple of A. Wherefore, by the preceding case, D is the same multiple of C; that is, C is the same part of D that A is of B. Therefore, if the first, &c. Q. E. D.

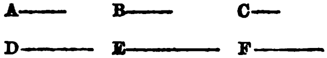
This proposition is the converse of the preceding one.

PROP. VII. THEOREM.

Equal magnitudes have the same ratio to a magnitude of the same kind: and conversely, a magnitude has the same ratio to equal magnitudes of the same kind.

Let A and B be equal magnitudes, and C any other of the same kind. First, the magnitudes A and B have each the same ratio to C.

Take of A and B any equimultiples whatever D and E, and of C any multiple whatever F.



Because D is the same multiple of A, that E is of B (*Const.*), and A is equal to B (*Hyp.*). Therefore D is equal to E (V. *Ax.* 1). Wherefore, if D be greater than F, E is greater than F; if equal, equal; and if less, less. But D and E are any equimultiples of A and B (*Const.*), and F is any multiple of C. Therefore, as A is to C, so is B to C (V. *Def.* 5).

Secondly, the magnitude C has the same ratio to A that it has to B.

For the same construction being made, it may be shown as above, that D is equal to E. Therefore if F be greater than D, F is likewise greater than E; if equal, equal; and if less, less. But F is any multiple whatever of C, and D and E are any equimultiples whatever of A and B. Therefore C is to A as C is to B (V. *Def.* 5). Therefore, equal magnitudes, &c. Q. E. D.

PROP. VIII. THEOREM.

Of unequal magnitudes, the greater has a greater ratio to a magnitude of the same kind, than the less has: and conversely, a magnitude has a greater ratio to the less of unequal magnitudes of the same kind, than it has to the greater.

Let A B, B C be two unequal magnitudes, of which A B is the greater, and let D be any other magnitude of the same kind.

First, the greater A B has a greater ratio to D, than B C has to D.

Fig. 1.

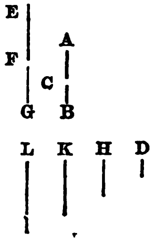


Fig. 2.

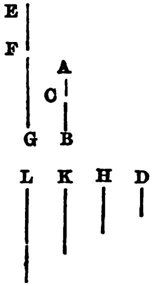
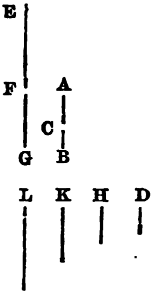


Fig. 3.



If the magnitude which is not the greater of the two A C and C B, be not less than D (as in fig. 1), take E F and F G, the doubles of A C and C B. But, if that which is not the greater of the two A C and C B, be less than D (as in figs. 2 and 3), take equimultiples of A C and C B,—viz., E F and F G, each greater than D. In all the cases, take H the double of D, K its triple, and so on, till L the multiple of D be found which first becomes greater than F G; and K the multiple of D which is next less than L, or the next preceding which is not greater than F G: that is, F G is not less than K.

Because E F is the same multiple of A C, that F G is of C B (*Const.*). Therefore F G is the same multiple of C B that E G is of A B (V. 1); that is, E G and F G are equimultiples of A B and C B. But F G is not less than K, and (*Const.*) E F is greater than D. Therefore the whole E G is greater than K and D together. But K together with D is equal to L (*Const.*). Therefore E G is greater than L. But F G is not greater than L (*Const.*); and E G and F G were proved to be equimultiples of A B and B C; and L is a multiple of D (*Const.*). Therefore A B has to D a greater ratio than B C has to D (V. *Def.* 7).

Secondly, D has to the less B C a greater ratio than it has to A B.

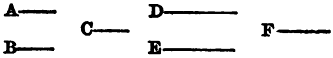
For, the same construction being made, it may be shown, as above, that L is greater than F G, but not greater than E G; and L is a multiple of D (*Const.*). And F G and E G were proved to be equimultiples of C B and A B. Therefore D has to C B a greater ratio than it has to A B (V. *Def.* 7). Wherefore, of unequal magnitudes, &c. Q. E. D.

PROP. IX. THEOREM

Magnitudes which have the same ratio to a magnitude of the same kind, are equal to one another: and conversely, magnitudes to which a magnitude of the same kind has the same ratio, are equal to one another.

Let the magnitudes A and B have each the same ratio to a magnitude *C of the* same kind. The magnitude A is equal to the magnitude *B.*

For if A be not equal to B, one of them must be greater than the other. Let A be the greater. As in the preceding proposition, take D and E, equimultiples of A and B, and F a multiple of C, such that D may be greater than F, but E not greater than F.



Because A is to C as B is to C (*Hyp.*), and of A and B, are taken equimultiples D and E, and of C is taken a multiple F. But D is greater than F (*Const.*). Therefore E is also greater than F (V. *Def.* 5). But E is not greater than F (*Const.*); which is impossible. Therefore A and B are not unequal; that is, they are equal.

Next, let the magnitude C have the same ratio to each of the magnitudes A and B. The magnitude A is equal to the magnitude B.

For, if A be not equal to B, one of them must be greater than the other; let A be the greater. As in the preceding proposition, take E and D equimultiples of B and A, and F a multiple of C, such that F may be greater than E, but not greater than D.

Because C is to B, as C is to A (*Hyp.*), and F the multiple of the first, is greater than E the multiple of the second. Therefore F, the multiple of the third, is greater than D the multiple of the fourth (V. *Def.* 5). But F is not greater than D (*Hyp.*); which is impossible. Therefore A and B are not unequal; that is, they are equal. Wherefore magnitudes which, &c. Q. E. D.

PROP. X. THEOREM.

Of two magnitudes, that which has the greater ratio to another magnitude of the same kind, is the greater of the two; and that magnitude to which another magnitude of the same kind has the greater ratio, is the less of the two.

Let A have to C a greater ratio than B has to C. A is greater than B.

As, in the preceding proposition, take D and E equimultiples of A and B, and F a multiple of C, such, that D is greater than F, but E not greater than F (V. *Def.* 7).

Because D and E are equimultiples of A and B, and D is greater than E (*Const.*). Therefore A is greater than B (V. *Ax.* 4).

Next, let C have a greater ratio to B than to A. B is less than A.

Take a multiple F of C, and equimultiples of E and D, of B and A, such that F is greater than E, but not greater than D (V. *Def.* 7).

Because E and D are equimultiples of B and A, and E is less than D (*Const.*). Therefore B is less than A (V. *Ax.* 4). Therefore, of two magnitudes, &c. Q. E. D.

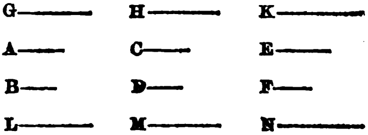
PROP. XI. THEOREM.

Ratios that are equal to the same ratio, are equal to one another.

If A is to B as C is to D; and C is to D, as E is to F. A is to B, as E is to F.

Take *of A, C,* and E, any eqimultiples whatever, G, H, and K: and *of B, D,* and F any equimultiples whatever, L, M, and N.

Because A is to B as C is to D, and G and H are taken equimultiples of A and C; and L and M, of B and D. If G be greater than L, H is greater than M; if equal, equal; and if less, less (V. *Def.* 5). Again, because C is to D, as E is to F, and H and K are taken equimultiples of C and E; and M and N, of D and F. If H be greater than M, K is greater than N; if equal, equal; and if less, less. But if G be greater than L, it has been shown that H is greater than M; if equal, equal; and if less, less. Therefore, if G be greater than L, K is greater than N; if equal, equal; and if less, less. But G and K are any equimultiples whatever of A and E; and L and N any whatever of B and F. Therefore, A is to B as E is to F (V. *Def.* 5). Wherefore, ratios that, &c. Q. E. D.



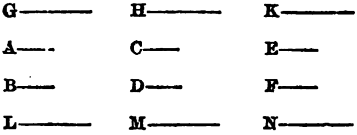
PROP. XII. THEOREM.

If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so are all the antecedents taken together to all the consequents.

Let any number of magnitudes A, B, C, D, E, and F, be proportionals; that is, as A is to B, so is C to D; and as C is to D, so is E to F. A is to B, as A, C, and E together, is to B, D, and F together.

Take of A, C, and E any equimultiples whatever G, H, and K; and of B, D, and F any equimultiples whatever, L, M, and N.

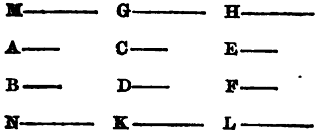
Because A is to B, as C is to D; and C is to D as E is to F; and that G, H, and K are equimultiples of A, C, and E; and L, M, and N; equimultiples of B, D, and F. Therefore, if G be greater than L, H is greater than M, and K greater than N; if equal, equal; and if less, less (V. *Def.* 5). Wherefore if G be greater than L, then G, H, and K together, are greater than L, M, and N together; if equal, equal; and if less, less. But if there be any number of magnitudes equimultiples of as many others, each of each, whatever multiple one of them is of its part, the same multiple is the whole of the whole (V. 1). Therefore G, and G, H, and K together, are any equimultiples of A, and A, C, and E together. For the same reason L, and L, M, and N together, are any equimultiples of B, and B, D, and F together. Therefore A is to B, as A, C, and E together, are to B, D, and F together (V. *Def.* 5). Wherefore, if any number, &c. Q. E. D.



PROP. XIII. THEOREM.

If the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth; the first has also to the second a greater ratio than the fifth has to the sixth.

Let A the first, have the same ratio to B the second, which C the third, has to D the fourth; but C the third a greater ratio to D the fourth than E the fifth, has to F the sixth.



The first A has to the second B, a greater ratio than the fifth E, has to the sixth F.

Take G and H equimultiples of C and E, and K and L equimultiples of D and F, such that G may be greater than K, but H not greater than L (V. *Def.* 7). Whatever multiple G is of C, take M the same multiple of A; and whatever multiple K is of D, take N the same multiple of B.

Because A is to B, as C is to D (*Hyp.*), and M and G are equimultiples of A and C; and N and K are equimultiples of B and D. Therefore, if M be greater than N, G is greater than K; if equal, equal; and if less, less (V. *Def.* 5). But G is greater than K (*Const.*). Therefore M is greater than N. But H is not greater than L (*Const.*); and M and H are equimultiples of A and E; and N and L equimultiples of B and F. Therefore A has a greater ratio to B, than E has to F (V. *Def.* 7). Wherefore, if the first, &c. Q. E. D.

COROLLARY.—If the first have a greater ratio to the second, than the third has to the fourth, but the third the same ratio to the fourth, which the fifth has to the sixth; it may be demonstrated, in like manner, that the first has a greater ratio to the second, than the fifth has to the sixth.

PROP. XIV. THEOREM.

If the first has the same ratio to the second which the third has to the fourth; and if the first be greater than the third, the second is greater than the fourth; if equal, equal; and if less, less.

Let the first A have the same ratio to the second B, which the third C, has to the fourth D.

Fig. 1.

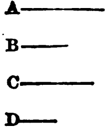


Fig. 2.

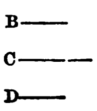
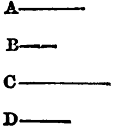


Fig. 3.



First, if A be greater than C (fig. 1), B is greater than D.

Because A is greater than C, and B is another magnitude of the same kind, A has to B a greater ratio than C has to B (V. 8). But, as A is to B, so is C to D (*Hyp.*). Therefore also C has to D a greater ratio than C has to B (V. 13). But of two magnitudes, that to which another of the same kind has the greater ratio is the less (V. 10). Therefore D is less than B; that is, B is greater than D.

Secondly, if A be equal to C (fig. 2), B is equal to D.

For A is to B, as C, that is, A is to D. Therefore B is equal to D (V. 9).

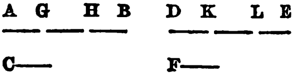
Thirdly, if A be less than C (fig. 3), B is less than D.

For C is greater than A, and C is to D, as A is to B. Therefore *D is greater than B,* by the first case; that is, B is less than D. Therefore, *if the first, &c.* Q. E. D.

PROP. XV. THEOREM.

Magnitudes have the same ratio to one another which their equimultiples have.

Let A B be the same multiple of C, that D E is of F. C is to F, as A B is to D E.



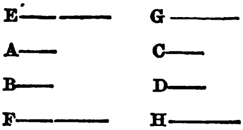
Divide A B into magnitudes, each equal to C, viz., A G, G H, and H B; and D E into as many (*Hyp.*) magnitudes, each equal to F, viz., D K, K L, and L E.

Because the magnitudes A G, G H, H B, are all equal to one another, and the magnitudes D K, K L, L E, are also equal to one another. Therefore A G is to D K as G H to K L, and as H B to L E (V. 7). But as one of the antecedents is to its consequent, so are all the antecedents together to all the consequents together (V. 12). Therefore, as A G is to D K, so is A B to D E. But A G is equal to C, and D K to F (*Const.*). Therefore as C is to F, so is A B to D E. Therefore, magnitudes, &c. Q. E. D.

PROP. XVI. THEOREM.

If four magnitudes of the same kind be proportionals, they are also proportionals when taken alternately.

Let A, B, C, and D be four magnitudes of the same kind, and let A be to B, as C is to D. They are also proportionals when taken alternately (*Def.* 13); that is, A is to C, as B to D.



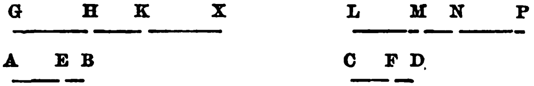
Take of A and B, any equimultiples whatever, E and F; and of C and D any equimultiples whatever, G and H.

Because E is the same multiple of A, that F is of B, and that magnitudes have the same ratio to one another which their equimultiples have (V. 15). Therefore A is to B as E is to F. But as A is to B, so is C to D (*Hyp.*). Therefore as C is to D, so is E to F (V. 11). Again, because B and H are equimultiples of C and D. Therefore as C is to D, so is G to H (V. 15). But it was proved that as C is to D, so is E to F. Therefore, as E is to F, so is G to H (V. 11). But when four magnitudes are proportionals, if the first be greater than the third, the second is greater than the fourth: if equal, equal; if less, less (V. 14). Therefore, if E be greater than G, F likewise is greater than H; if equal, equal; and if less, less. But E and F are any equimultiples whatever of A and B (*Const.*) and G and H any whatever of C and D. Therefore A is to C as B to D. (V. *Def.* 5.) If then four magnitudes, &c. Q. E. D.

PROP. XVII. THEOREM.

If four magnitudes be proportionals; by division the excess of the first above the second is to the second, as the excess of the third above the fourth is to the fourth.

Let A B, B E, C D, and D F, be proportionals; that is, as A B to B E, so is CD to DF. And let AE be the excess of AB above BE; and CF the excess of CD above DF. As AE is to EB, so is CF to FD (V. *Def.* 16).



Take of AE, EB, CF, and FD any equimultiples whatever GH, HK, LM, and MN; and of EB and FD any other equimultiples whatever KX and NP.

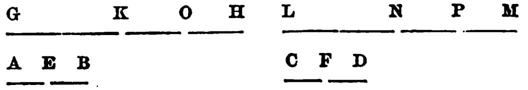
Because GH is the same multiple of AE, that HK is of EB (*Const.*). Therefore, GH is the same multiple of AE, that GK is of AB (V. 1). But GH is the same multiple of AE that LM is of CF (*Const.*). Therefore GK is the same multiple of AB, that LM is of CF. Again, because LM is the same multiple of CF, that MN is of FD (*Const.*). Therefore LM is the same multiple of CF, that LN is of CD (V. 1). But LM was shown to be the same multiple of CF, that GK is of AB. Therefore GK is the same multiple of AB, that LN is of CD; that is, GK and LN are equimultiples of AB and CD. Next, because HK is the same multiple of EB, that MN is of FD (*Const.*), and KX the same multiple of EB, that NP is of FD (*Const.*). Therefore HX is the same multiple of EB, that MP is of FD (V. 2). Because AB is to BE as CD is to DF (*Hyp.*), and GK and LN are equimultiples of AB and CD, and HX and MP are equimultiples of EB and FD. Therefore if GK be greater than HX, LN is greater than MP; if equal, equal; and if less, less (V. *Def.* 5). But if GH be greater than KX, by adding the common part HK to these unequals, GK is greater than HX (I. *Ax.* 4). Wherefore also LN is greater than MP. By taking away MN from these unequals, LM is greater than NP (I. *Ax.* 5). Therefore, if GH be greater than KX, LM is greater than NP. In like manner it may be shown that if GH be equal to KX, LM is equal to NP; and if less, less: but GH and LM are any equimultiples whatever of AE and CF (*Const.*), and KX and NP are any whatever of EB and FD. Therefore, AE is to EB, as CF is to FD (V. *Def.* 5). If then magnitudes, &c. Q. E. D.

The term *division* used in the enunciation of this proposition, is not used in the arithmetical sense of that term, but in that of separation or subtraction.

PROP. XVIII. THEOREM.

If four magnitudes be proportionals; by composition, the first and second together are to the second, as the third and fourth together are to the fourth.

Let AE, EB, CF, and FD be proportionals; that is, as AE to EB, so let CF be to FD. And, let AB be the sum of AE and EB; and



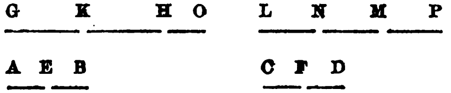
CD the sum of CF and FD. As AB to BE, so is CD to DF (V. Def. *15*).

Take of A B, B E, C D, and D F any equimultiples whatever G H, H K, L M, and M N; and of B E, D F, any other equimultiples whatever K O and N P.

Because K O and N P are equimultiples of B E and D F; and K H and N M are likewise equimultiples of B E and D F. If K O, the multiple of B E, be greater than K H, which is also a multiple of B E. Therefore, N P, the multiple of D F, is also greater than N M, the multiple of the same D F; if K O be equal to K H, N P is equal to N M; and if less, less.

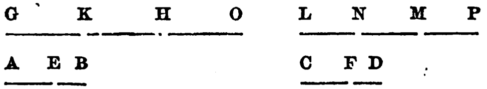
First, if K O be not greater than K H; N P is not greater than N M. Because, G H and H K are equimultiples of A B and B E, and A B is greater than B E. Therefore G H is greater than H K (*Ax.* 3); but K O is not greater than K H (*Hyp.*). Therefore G H is greater than K O. In like manner it may be shown, that L M is greater than N P. Therefore if K O be not greater than K H, G H, the multiple of A B, is greater than K O, the multiple of B E; and likewise L M, the multiple of C D, is greater than N P, the multiple of D F.

Next, let K O be greater than K H. Therefore, as has been shown, N P is greater than N M. Because the whole G H is the same multiple of the whole A B, that H K is of B E. Therefore the remainder G K is the same multiple of the remainder A E that G H is of A B (V. 5), which is the same that L M is of C D. In like manner, because L M is the same multiple of C D, that M N is of D F. Therefore the remainder L N is the same multiple of the remainder C F, that the whole L M is of the whole C D (V. 5). But it was shown that L M is the same multiple of C D, that G K is of A E. Therefore G K is the same multiple of A E, that L N is of C F; that is, G K and L N are equimultiples of A E and C F. But K O and N P are equimultiples of B E and D F; and if from K O and N P there be taken H K and M N, which are likewise equimultiples of B E and D F. Therefore, the remainders H O and M P are either equal to B E and D F, or equimultiples of them (V. 6).



First, let H O and M P be equal to B E and D F. Because A E is to E B, as C F to F D (*Hyp.*), and G K and L N are equimultiples of A E and C F. Therefore G K is to E B, as L N to F D (V. 4, *Cor.*). But H O is equal to E B, and M P to F D. Therefore G K is to H O, as L N to M P. Wherefore if G K be greater than H O, L N is greater than M P; if equal, equal; and if less, less (V. A).

Next, let H O and M P be equimultiples of E B and F D. Because



A E is to E B, as C F to F D (*Hyp.*), and of A E and C F are taken *equimultiples G K* and L N; and of E B and F D, the equimultiples

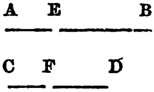
H O and M P. If G K be greater than H O, L N is greater than M P; if equal, equal; and if less, less (V. *Def.* 5); which was likewise shown in the preceding case. But if G H be greater than K O, taking K H from both, G K is greater than H O (I. *Ax.* 5). Therefore also L N is greater than M P. By adding N M to these unequals, L M is greater than N P (I. *Ax.* 4). Therefore, if G H be greater than K O, L M is greater than N P. In like manner it may be shown, that if G H be equal to K O, L M is equal to N P; and if less, less. But in the case in which K O is not greater than K H, it has been shown that G H is always greater than K O, and likewise L M greater than N P. And G H and L M are any equimultiples whatever of A B and C D (*Const.*), also K O and N P are any whatever of B E and D F. Therefore, as A B is to B E, so is C D to D F (V. *Def.* 5). If four magnitudes, &c. Q. E. D.

The term *composition* used in the enunciation of this proposition, signifies simply the addition of the two magnitudes, or the finding of one magnitude equal to both.

PROP. XIX. THEOREM.

If there be two magnitudes such that the first is to the second, as a part of the first is to a part of the second; the remainder is to the remainder as the first is to the second.

Let A B be to C D, as A E a part of A B is to C F a part of C D. The remainder E B is to the remainder F D, as A B is to C D.



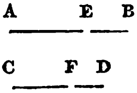
Because A B is to C D, as A E to C F. Therefore alternately, B A is to A E, as D C to C F (V. 16). Because E B is the excess of A B above A E, and D F the excess of C D above C F. Therefore, as B E is to E A, so is D F to F C (V. 17). But alternately, as B E is to D F, so is E A to F C (V. 16), and as A E to C F, so is A B to to C D (*Hyp.*). Therefore B E is to D F, as A B is to C D (V. 11). Wherefore, if there be two magnitudes such, &c. Q. E. D.

COROLLARY.—If there be two magnitudes such that the first is to the second as a part of the first is to a part of the second; the remainder is to the remainder, as the part of the first is to the part of the second. The demonstration is contained in the preceding.

PROP. E. THEOREM.

If four magnitudes be proportionals, they are also proportionals by conversion; that is, the first is to its excess above the second, as the third to its excess above the fourth.

Let A B be to B E, as C D to D F. And let A E be the excess of A B above B E, and C F the excess of C D above D F. A B is to A E, as D C to C F. Because A B is to B E, as C D to D F (*Hyp.*). Therefore, by division, A E is to E B, as C F to F D (V. 17). But, by inversion, B E is to E A, as D F to F C (V. B). Wherefore, by composition, B A is to A E, as D C is to C F (V. 18). If therefore four, &c. Q. E. D.



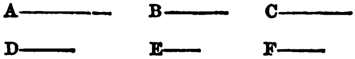
This proposition was added by Dr. Simson, as a substitute for a corollary to the *next* proposition given in the original Greek, which he declares to be vitiated.

PROP. XX. THEOREM.

If there be three magnitudes, and other three, which, taken two and two, have the same ratio; and if the first be greater than the third, the fourth is greater than the sixth; if equal, equal; and if less, less.

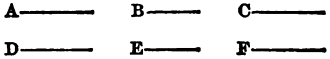
Let A, B, and C be three magnitudes, and D, E, and F other three, which taken two and two have the same ratio; viz., as A is to B, so is D to E; and as B is to C, so is E to F. If A be greater than C, D is greater than F; if equal, equal; and if less, less.

First, let A be greater than C. D is greater than F.



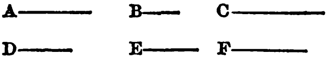
Because A is greater than C, and B is any other magnitude; and, of unequal magnitudes, the greater has a greater ratio to a magnitude of the same kind than the less has (V. 8). Therefore A has to B a greater ratio than C has to B. But as D is to E, so is A to B (*Hyp.*). Therefore D has to E a greater ratio than C has to B (V. 13). Because B is C, as E to F (*Hyp.*). Therefore, by inversion, C is to B, as F is to E (V. B). But D was shown to have to E a greater ratio than C has to B. Therefore D has to E a greater ratio than F has to E (V. 13, *Cor.*). But of two unequal magnitudes that which has a greater ratio to another magnitude of the same kind is the greater of the two (V. 10). Therefore D is greater than F.

Secondly, let A be equal to C. D is equal to F.



Because A and C are equal to one another, A is to B, as C is to B (V. 7). But A is to B, as D to E (*Hyp.*), and C is to B, as F to E (*Hyp.*). Therefore D is to E, as F to E (V. 11, and V. B). Wherefore D is equal to F (V. 9).

Next, let A be less than C. D is less than F.



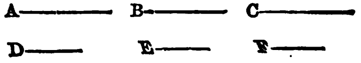
For C is greater than A; and as was shown in the first case, C is to B, as F to E. In like manner, B is to A, as E to D. Therefore F is greater D, by the first case; that is, D is less than F. Therefore, if there be three, &c. Q. E. D.

PROP. XXI. THEOREM.

If there be three magnitudes, and other three, which have the same ratio taken two and two, but in a cross order; and if the first magnitude be greater than the third, the fourth is greater than the sixth; if equal, equal; if less, less.

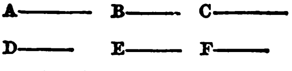
Let A, B, and C be three magnitudes, and D, E, and F other three, which have the same ratio, taken two and two, but in a cross order,—viz., as A is to B, so is E to F, and as B is to C, so is D to E. If A be greater than C, D is greater than F; if equal, equal; and if less, less.

First, let A be greater than C. *D is greater than F.*



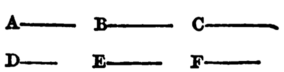
Because A is greater than C, *and B is any other* magnitude, A has to B a greater ratio than C has to B (V. 8). But as E to F, so is A to B (*Hyp.*). Therefore E has to F a greater ratio than C has to B (V. 13). Because B is to C, as D is to E (*Hyp.*). Therefore, by inversion, C is to B, as E is to D (V. B). But E was shown to have to F a greater ratio than C has to B. Therefore E has to F a greater ratio than E has to D (V. 13, *Cor.*) But of two unequal magnitudes that to which another magnitude of the same kind has a greater ratio is the less of the two (V. 10). Therefore F is less than D; that is, D is greater than F.

Secondly, let A be equal to C. D is equal to F.



Because A and C are equal, A is to B, as C is to B (V. 7). But A is to B, as E is to F (*Hyp.*), and C is to B as E is to D. Therefore, E is to F, as E is to D (V. 11). Wherefore D is equal to F (V. 9).

Next, let A be less than C. D is less than F.

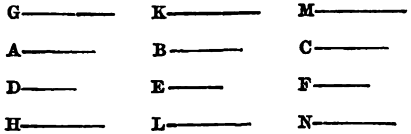


For C is greater than A; and, as was shown, C is to B, as E is to D. In like manner B is to A, as F is to E. Therefore F is greater than D, by case first; that is, D is less than F. Therefore, if there be three, &c. Q. E. D.

PROP. XXII. THEOREM.

If there be any number of magnitudes, and as many others, which taken two and two in order, have the same ratio; the first shall have to the last of the first magnitudes, the same ratio which the first has to the last of the others.

First, let there be three magnitudes, A, B, and C, and as many others D, E, and F, which taken two and two in order, have the same ratio, that is, such that A is to B as D is to E; and as B is to C, so is E to F. A is to C, as D is to F.



Take of A and D, any equimultiples whatever G and H; of B and E, any equimultiples whatever K and L; and of C and F any whatever M and N.

Because A is to B, as D is to E, and G and H are equimultiples of A and D, and K and L equimultiples of B and E. Therefore as G is to K, so is H to L (V. 4). For the same reason, K is to M as L is to N. Because there are three magnitudes G, K, and M, and other three H, L, and N, which two and two, have the same ratio. Therefore if G be greater than M, H is greater than N; if equal, equal; and if less, less (V. 20). But G and H are any equimultiples whatever of A and D, and M and N are any equimultiples whatever of C and F (*Const.*). Therefore, as A is to C, so is D to F (V. *Def.* 5).

Next, let there be four magnitudes, A, B, C, and D, and other four E, F, G, and H, which two and two have the same ratio,—viz., as A is to B, *so is E to F; as* B is to C, so is F to G; and as C is to D, so is G to H. *A is to D, as* E is to H

Because A, B and C are three magnitudes, and E, F and G other three, which taken two and two, have the same ratio. Therefore by the foregoing case, A is to C, as E is to G. But C is to D, as G is to H. Therefore again, by the first case, A is to D, as E is to H; and so on, whatever be the number of magnitudes. Therefore, if there be any number, &c. Q. E. D.

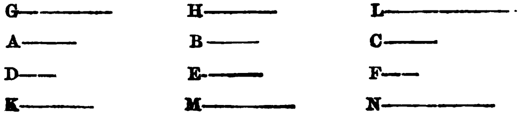
A.B.C.D
E.F.G.H

N.B. This proposition is usually cited by the words "*ex æquali*," or "*ex æquo*," that is, from equality of distance (*Def.* 19).

PROP. XXIII. THEOREM.

If there be any number of magnitudes, and as many others, which taken two and two in a cross order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first has to the last of the others.

First, let there be three magnitudes A, B and C, and other three D, E and F, which taken two and two in a cross order have the same ratio; that is, such that A is to B, as E is to F; and as B is to C, so is D to E. A is to C, as D is to F.



Take of A, B and D any equimultiples whatever G, H and K; and of C, E and F any equimultiples whatever, L, M and N.

Because G and H are equimultiples of A and B, and magnitudes have the same ratio which their equimultiples have (V. 15). Therefore as A is to B, so is G to H. For the same reason, as E is to F, so is M to N. But as A is to B, so is E to F (*Hyp.*). Therefore as G is to H, so is M to N (V. 11). Because as B is to C, so is D to E (*Hyp.*), and H and K are equimultiples of B and D, and L and M of C and E. Therefore as H is to L, so is K to M (V. 4). But it has been shown that G is to H, as M is to N. Therefore, there are three magnitudes G, H and L, and other three K, M and N, which have the same ratio taken two and two in a cross order; and if G be greater than L, K is greater than N: if equal, equal; and if less, less (V. 21). But G and K are any equimultiples whatever of A and D (*Const.*); and L and N any whatever of C and F. Therefore as A is to C, so is D to F (V. *Def.* 5).

Next, let there be four magnitudes A, B, C and D, and other four E, F, G and H, which taken two and two in a cross order have the same ratio, viz., A to B, as G to H; B to C, as F to G; and C to D, as E to F. A is to D, as E is to H.

Because A, B and C are three magnitudes, and F, G and H other three, which taken two and two in a cross order, have the same ratio. Therefore, by the first case, A is to C, as F is to H. But C is to D, as E is to F. Wherefore again, by the first case, A is to D, as E is to H; and so on, whatever be the number of magnitudes. Therefore, if there be any number, &c. Q. E. D.

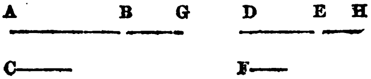
A.B.C.D
E.F.G.H

N.B. This proposition is usually cited by the words "*ex æquali in proportione perturbatâ*;" or "*ex æquo perturbato*, that is, in disorderly proportion.

PROP. XXIV. THEOREM.

If the first has to the second the same ratio which the third has to the fourth; and the fifth to the second the same ratio which the sixth has to the fourth; the first and fifth together have to the second, the same ratio which the third and sixth together have to the fourth.

Let A B the first, have to C the second, the same ratio which D E the third, has to F the fourth; and let B G the fifth, have to C the second, the same ratio which E H the sixth, has to F the fourth. A G, the first and fifth together, have to C the second, the same ratio which D H, the third and sixth together, have to F the fourth.



Because B G is to C, as E H is to F. Therefore by inversion, C is to B G, as F is to E H (V. B). Because, as A B is to C, so is D E to F (*Hyp.*), and as C is to B G, so is F to E H. Therefore, *ex æquali*, A B is to B G, as D E to E H (V. 22). Because A G is the sum of A B and B G, and D H the sum of D E and E H. Therefore as A G is to G B, so is D H to H E (V. 18). But as G B is to C, so is H E to F (*Hyp.*). Therefore, *ex æquali*, as A G is to C, so is D H to F (V. 22). Wherefore, if the first, &c. Q. E. D.

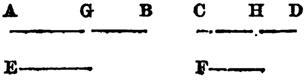
Corollary 1.—If the same hypothesis be made as in the proposition, the excess of the first and fifth is to the second, as the excess of the third and sixth is to the fourth. The demonstration of this is the same as that of the proposition, if division be used instead of composition.

Corollary 2.—The proposition holds true of two ranks of magnitudes, whatever be their number, of which each of the first rank has to the second, the same ratio that the corresponding one of the second rank has to a fourth magnitude.

PROP. XXV. THEOREM.

If four magnitudes of the same kind be proportionals, the greatest and least of them together are greater than the other two together.

Let the four magnitudes A B, C D, E and F be proportionals; that is A B is to C D, as E to F. And let A B be the greatest of them, and consequently F the least (V. 14, and A). A B and F are together greater than C D and E together.



Take A G equal to E, and C H equal to F.

Because A B is to C D, as E is to F, and A G is equal to E, and C H equal to F. Therefore A B is to C D, as A G is to C H (V. 11, and 7). Because A B is to C D, as A G is to C H, the remainder G B is to the remainder H D, as A B is to C D (V. 19). But A B is greater than C D (*Hyp.*). Therefore G B is greater than H D (V. A). Because A G is equal to E, and C H to F. Therefore, A G and F together, are equal to C H and E together (I. *Ax.* 2). If to the unequal magnitudes G B and H D, of *which G B is* the greater, there be added equal magnitudes,—viz., to G B the two AG and F, and to HD, the two CH and E. Therefore AB and F together are greater than CD and E together (I. *Ax.* 4). Therefore, if four magnitudes, &c. Q. E. D.

Corollary.—The sum of the extremes of three continual proportionals is greater than double the mean.

The following "propositions," says Dr. Simson, "are annexed to the fifth Book because they are frequently made use of, both by ancient and modern geometers: and in many cases, compound ratios cannot be brought into demonstration, without making use of them."

PROP. F. THEOREM.

Ratios compounded of equal ratios, are equal to one another.

Let A be to B, as D is to E; and B to C, as E is to F. The ratio compounded of the ratios of A to B, and of B to C, which (*Def.* A) is the ratio of A to C, is equal to the ratio which is compounded of the ratios of D to E, and of E to F, which is the ratio of D to F.

A.B.C
D.E.F

Because there are three magnitudes, A, B and C, and other three D, E and F, which, taken two and two, in order, have the same ratio. Therefore *ex æquali*, A is to C, as D is to F (V. 22).

Next, let A be to B, as E is to F, and B to C, as D is to E.

Because there are three magnitudes A, B and C, and other three D, E and F, which taken two and two, in a cross order, have the same ratio. Therefore, *ex æquali in proportione perturbata*, A is to C, as D is to F (V. 23). Wherefore, the ratio of A to C, which is compounded of the ratios of A to B, and B to C, is the same with the ratio of D to F, which is compounded of the ratios of D to E, and E to F.

A.B.C
D.E.F

And in like manner the proposition may be demonstrated, whatever be the number of ratios in either case. Q. E. D.

PROP. G. THEOREM.

If several ratios be equal to several other ratios, each to each; the ratio compounded of ratios which are equal to the first ratios, each to each, are equal to the ratio compounded of ratios which are equal to the other ratios, each to each.

If A is to B, as E is to F; and C is to D, as G is to H; also, if A is to B, as K is to L; and C is to D, as L is to M. Moreover, if E is to F, as N is to O; and G is to H, as O is to P. K is to M, as N is to P.

The ratio of K to M (*Def.* A) is compounded of the ratios of K to L, and of L to M, which are equal (*Hyp.*) to the ratios of A to B, and of C to D. The ratio of N to P is compounded of the ratios of N to O, and of O to P, which are equal (*Hyp.*) to the ratios of E to F and of F to G.

A.B.C.D.K.L.M
E.F.G.H.N.O.P

Because K is to L, as A is to B, that is, as E to F, that is as N to O (*Hyp.*); and, L is to M, as C is to D, that is, as G is to H, that is, as O is to P. Therefore, *ex æquali*, K is to M, as N is to P (V. 22).

Therefore, if several ratios, &c. Q. E. D

PROP. H. THEOREM.

If a ratio which is compounded of several ratios be equal to a ratio which is compounded of several other ratios; and if one of the first ratios, or the ratio which is compounded of several of them, be equal to one of the last ratios, or to the ratio which is compounded of several of them; the remaining ratio of the first, or, if there be more than one, the ratio compounded of the remaining ratios, is equal to the remaining ratio of the last, or, if there be more than one, to the ratio compounded of these remaining ratios.

Let the first ratios be those of A to B, B to C, C to D, D to E, and E to F; and let the other ratios be those of G to H, H to K, K to L, and L to M. Also let the ratio of A to F, compounded of the first ratios, be equal to the ratio of G to M, compounded of the other ratios. Besides, let the ratio of A to D, compounded of the ratios of A to B, B to C, and C to D, be equal to the ratio of G to K, compounded of the ratios of G to H, and H to K.

The ratio compounded of the remaining first ratios, viz., of the ratios of D to E, and E to F, which compounded ratio is the ratio of D to F, is equal to the ratio of K to M, which is compounded of the remaining ratios of K to L, and L to M of the other ratios.

A.B.C.D.E.F
G.H.K.L.M

Because (*Hyp.*) A is to D, as G is to K. Therefore, by inversion, D is to A, as K to G (V. B.). Because A is to F, as G is to M (*Hyp.*). Therefore, *ex æquali*, D is to F, as K is to M (V. 22). If, therefore, a ratio which is, &c. Q. E. D.

PROP. K. THEOREM.

If there be any number of ratios, and any number of other ratios such, that the ratio compounded of ratios equal to the first ratios, each to each, is equal to the ratio compounded of ratios which are equal, each to each, to the last ratios; and if one of the first ratios, or the ratio which is compounded of ratios equal to several of the first ratios, each to each, be equal to one of the last ratios, or to the ratio compounded of ratios which are equal, each to each, to several of the last ratios; the remaining ratio of the first, or, if there be more than one, the ratio compounded of ratios equal, each to each, to the remaining ratios of the first, is equal to the remaining ratio of the last, or, if there be more than one, to the ratio compounded of ratios which are equal, each to each, to these remaining ratios.

Let the ratios of A to B, C to D, and E to F, be the first ratios: and the ratios of G to H, K to L, M to N, O to P, Q to R, be the other ratios: and let A be to B, as S to T; and C to D, as T to V; and E to F, as V to X: and therefore (*Def.* A), the ratio of S to X is compounded of the ratios of S to T, T to V, and V to X, which are equal to the ratios of A to B, C to D, and E to F: each to each.

Also, let G be to H, as Y to Z: and K to L as Z to *a*; M to N as *a* to *b*; O to P, as *b* to *c*; and Q to R, as *c* to *d*: and therefore (*Def.* A), the ratio of Y to *d* is compounded of the ratios of Y to Z, Z to a, a to b, b to *c, and c to d,* which are equal, each to each, to the ratios of G to H, K to *L, M to N, O to P,* and Q to R: wherefore (Hyp.), S is to X, as Y to d.

Also, let the ratio of A to B, that is, the ratio of S to T, which is one of the first ratios, be equal to the ratio of *e* to *g*, which is compounded of the ratios of *e* to *f*, and *f* to *g*, which (*Hyp.*) are equal to the ratios of G to H, and K to L, two of the other ratios; and let the ratio of *h* to *l* be that which is compounded of the ratios of *h* to *k*, and *k* to *l*, which are equal to the remaining first ratios, viz., of C to D, and E to F.

Also, let the ratio of *m* to *p*, be that which is compounded of the ratios of *m* to *n*, *n* to *o*, and *o* to *p*, which are equal, each to each, to the remaining other ratios, viz., of M to N, O to P, and Q to R.

The ratio of *h* to *l* is equal to the ratio of *m* to *p*; that is, *h* is to *l*, as *m* is to *p*.

h, k, l
A. B; C, D; E, F, S, T, V, X.
G, H; K, L; M, N; O, P; Q, R. Y, Z; a, b, c, d.
e, f, g. m, n, o, p.

Because *e* is to *f*, as G to H, that is, as Y to Z; and *f* is to *g*, as K to L, that is, as Z to *a*. Therefore, *ex æquali*, *e* is to *g*, as Y to *a* (V. 22). But A is to B, that is, S is to T, as *e* is to *g* (*Hyp.*). Therefore S is to T, as Y is to *a* (V. 11). Wherefore, by inversion, T is to S, as *a* is to Y (V. B). But S is to X, as Y is to D (*Hyp.*). Therefore, *ex æquali*, T is to X as *a* is to *d*. Because *h* is to *k*, as C is to D, that is, as T is to V (*Hyp.*); and *k* is to *l* as E is to F, that is, as V is to X. Therefore, *ex æquali*, *h* is to *l*, as T is to X. In like manner, it may be shown that *m* is to *p*, as *a* is to *d*. But it has been shown that T is to X, as *a* is to *d*. Therefore *h* is to *l*, as *m* is to *p* (V. 11). Wherefore, if there be any number of ratios, &c. Q. E. D.

"The propositions G and K are usually, for the sake of brevity, expressed in the same terms with propositions F and H: and therefore it was proper to show the true meaning of them when they are so expressed; especially since they are very frequently made use of by geometers."

After having laboured to correct this book of Euclid, Dr. Simson says, "I most readily agree with what the learned Dr. Barrow says, 'that there is nothing in the whole body of the Elements of a more subtile invention, nothing more solidly established, and more accurately handled, than the doctrine of proportionals.' And there is some ground to hope, that geometers will think that this could not have been said with as good reason, since Theon's time (A.D. 380), till the present." The modesty of this remark is only surpassed by its truth. Most editors, since Dr. Simson's time, have only rendered the doctrine of proportion more obscure; as a remarkable example, see Professor De Morgan's "Connection of Number and Magnitude."

BOOK VI.

DEFINITIONS.

I.

Similar rectilineal figures are those which have their several angle equal, each to each, and the sides about the equal angles proportionals.

In the case of triangles, this definition is redundant. For it is proved in Prop. IV. of this Book, that the sides about the equal angles of equiangular triangles are proportionals. In the case of quadrilaterals, or polygons, however, the definition is necessary. According to this definition, all equilateral triangles, squares, and regular polygons are similar rectilineal figures.

II.

Triangles and parallelograms are said to have their sides reciprocally proportional, when the sides about two of their angles are proportionals in such a manner, that a side of the first figure is to a side of the second, as the remaining side of the second is to the remaining side of the first.

Two magnitudes of any kind may be said to be reciprocally proportional to other two of the same kind, when one of the first pair is to one of the second pair, as the remaining one of the second pair is to the remaining one of the first.

III.

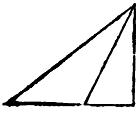
A straight line is said to be cut in extreme and mean ratio, when the whole is to the greater segment, as the greater segment is to the less.

A straight line is said to be cut in harmonical ratio, when the whole is to one of the extreme segments as the other extreme segment is to the middle segment. For brevity's sake, this mode of dividing a straight line is called *harmonical section*; and the mode of dividing a straight line explained in Euclid's definition is called *medial section*.

IV.

The altitude of any figure is the straight line drawn from its vertex perpendicular to the base.

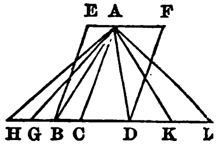
Altitude is a term synonymous with perpendicular. By the vertex of a figure here, is meant that angular point of the figure which is most remote from *any side* of the figure *assumed* at the base, the degree of remoteness being measured by a perpendicular drawn to that side or that side produced, from the said vertex; in other words, the longest perpendicular drawn from any angular point to the base, or base produced, is the altitude.



PROP. I. THEOREM.

Triangles and parallelograms of the same altitude are to one another as their bases.

Let the triangles A B C and A C D, and the parallelograms E C and C F, have the same altitude, viz., the perpendicular drawn from the point A to B D or B D produced. As the base B C is to the base C D, so is the triangle A B C to the triangle A C D; and as the base BC is to the base C D, so is the parallelogram E C to the parallelogram C F.



Produce B D both ways to the points H and L, and take any number of straight lines B G and G H, each equal to the base B C (I. 3); and D K and K L, any number of straight lines each equal to the base C D. Join A G, A H, A K, and A L.

Because C B, B G and G H, are all equal, the triangles A H G, A G B and A B C, are all equal (I. 38). Therefore, whatever multiple the base H C is of the base B C, the same multiple is the triangle A H C of the triangle A B C. For the same reason, whatever multiple the base L C is of the base C D, the same multiple is the triangle A L C of the triangle A D C. But if the base H C be equal to the base C L, the triangle A H C is also equal to the triangle A L C (I. 38); and if the base H C be greater than the base C L, the triangle A H C is likewise greater than the triangle A L C; and if less, less. Because there are four magnitudes, viz., the two bases B C and C D, and the two triangles A B C and A C D; and of the base B C, and the triangle A B C, the first and the third, any equimultiples whatever have been taken, viz., the base H C and the triangle A H C; and of the base C D, and the triangle A C D, the second and the fourth, any equimultiples whatever have been taken, viz., the base C L and the triangle A L C. And it has been shown, that, if the base H C be greater than the base C L, the triangle A H C is greater than the triangle A L C; if equal, equal; and if less, less. Therefore, as the base B C is to the base C D, so is the triangle A B C to the triangle A C D (V. *Def.* 5). Because the parallelogram C E is double of the triangle A B C (I. 41), and the parallelogram C F double of the triangle A C D, and magnitudes have the same ratio which their equimultiples have (V. 15). Therefore, as the triangle A B C is to the triangle A C D, so is the parallelogram E C to the parallelogram C F. But it has been shown, that, as the base B C is to the base C D, so is the triangle A B C to the triangle A C D. And as the triangle A B C is to the triangle A C D, so is the parallelogram E C to the parallelogram C F. Therefore, as the base B C is to the base C D, so is the parallelogram E C to the parallelogram C F (V. 11). Wherefore, triangles, &c. Q. E. D.

COROLLARY.—From this it is plain, that triangles and parallelograms that have equal altitudes, are to one another as their bases.

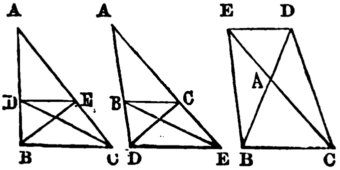
Let the figures be placed so as to have their bases in the same straight line; and having drawn perpendiculars from the vertices of the triangles to the bases, the straight line which joins the vertices is parallel to that in which their bases are (I. 33), because the perpendiculars are both equal and parallel to one another (I. 28). Then, if the same construction be made as in the proposition, the demonstration will be the same.

Exercise.—**Triangles and parallelograms upon equal bases are to one another as their altitudes.**

PROP. II. THEOREM.

If a straight line be drawn parallel to one of the sides of a triangle, it cuts the other two sides, or these sides produced, proportionally, so that the segments between the base and the parallel, are homologous; and conversely, if the two sides, or these sides produced, be cut proportionally, so that the segments between the base and the parallel are homologous, the straight line which joins the points of section is parallel to the base of the triangle.

Let D E be drawn parallel to B C, one of the sides of the triangle A B C. The sides A B and A C, or A B and A C produced, are cut proportionally; that is, B D is to D A, as C E is to E A.



Join B E and C D. The triangle B D E is equal to the triangle C D E (I. 37), because they are on the same base D E, and between the same parallels D E, and B C. But A D E is another triangle; and equal magnitudes have the same ratio to the same magnitude (V. 7). Therefore, as the triangle B D E is to the triangle A D E, so is the triangle C D E to the triangle A D E. But the triangle B D E is to the triangle A D E, as B D is to D A (VI. 1), because their altitude is the perpendicular drawn from the point E to A B, and they are to one another as their bases. For the same reason, the triangle C D E is to the triangle A D E, as C E to E A. Therefore, as B D is to D A, so is C E to E A (V. 11).

Next, let the sides A B and A C of the triangle A B C, or A B and A C produced, be cut proportionally in the points D and E, that is, so that B D is to D A as C E to E A. Join D E, and it is parallel to B C.

The same construction being made, because B D is to D A as C E is to E A. But B D is to D A, as the triangle B D E is to the triangle A D E (VI. 1); and C E is to E A, as the triangle C D E is to the triangle A D E. Therefore the triangle B D E is to the triangle A D E, as the triangle C D E is to the triangle A D E (V. 11). Wherefore the triangles B D E and C D E have the same ratio to the triangle A D E. Therefore the triangle B D E is equal to the triangle C D E (V. 9); and they are on the same base D E. But equal triangles on the same base and on the same side of it, are between the same parallels (I. 39). Therefore D E is parallel to B C. Wherefore, if a straight line, &c. Q. E. D.

The necessity for three diagrams in this proposition arises from the variety of position which may be given to the straight line drawn parallel to the base. This parallel may be drawn *between the vertex* of the triangle *and the base*, as in the first figure; *beyond the base*, as in the second figure, when the sides must be produced through the extremities of the base to meet it; or, *beyond the vertex*, as in the third figure, when the sides must be produced through the vertex, to meet it. In all these cases, the demonstration holds equally good, and it should be read with each figure separately, in order to render the argument clear to the mind. A more general method of ennunciating this proposition is contained in the corollary.

Corollary.—The triangles which two intersecting straight lines form with two parallel straight lines, have their sides on the intersecting lines proportionals, *and those in the same straight line are homologous*; and conversely, the two *straight lines*, which, with two intersecting straight lines, form triangles, having their sides on the intersecting straight lines proportionals, and those in the same straight line homologous, are parallel.

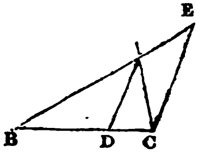
Exercise 1.—If a straight line be drawn parallel to the base and cutting the sides of of a triangle, these sides are proportional to the segments cut off each of them respectively; and in the two triangle thus formed, the sides about the common angle, or the vertical angles, are proportionals.

Exercise 2.—If several straight lines be drawn parallel to the base and cutting the sides of a triangle, the segments of the sides intercepted between the same parallels are proportional to each other, and to the sides from which they are respectively cut off.

PROP. III. THEOREM.

If any angle of a triangle be bisected by a straight line which also cuts the base, the segments of the base have the same ratio to one another which the adjacent sides of the triangle have; and conversely, if the segments of the base have the same ratio to one another which the adjacent sides of the triangle have, the straight line drawn from the vertex to the point of section bisects the vertical angle.

Let A B C be a triangle, and let the angle B A C be bisected by the straight line A D. The segment B D is to the segment D C, as the side B A is to the side A C.



Through the point C draw C E parallel to D A (I. 31); and let B A produced meet C E in E.

Because the straight line A C meets the parallels A D and E C, the angle A C E is equal to the alternate angle C A D (I. 29). But the angle C A D (*Hyp.*) is equal to the angle B A D. Therefore the angle B A D is equal to the angle A C E (I *Ax.* 1). Again, because the straight line B A E meets the parallels A D and E C, the exterior angle B A D is equal to the interior and opposite angle A E C (I. 29). But the angle A C E has been proved equal to the angle B A D. Therefore also the angle A C E is equal to the angle A E C (I. *Ax.* 1), and the side A E to the side A C (I. 6). Because A D is drawn parallel to E C, one of the sides of the triangle B C E. Therefore B D is to D C, as B A to A E (VI. 2). But A E is equal to A C. Therefore, B D is to D C, as B A is to A C (V. 7).

Next, let the segment B D be to the segment D C, as the side B A is to the side A C; and let A D be joined. The angle B A C is bisected by the straight line A D.

The same construction being made; because B D is to D C, as B A is to A C; and B D is to D C, as B A is to A E, because A D is parallel to E C (VI. 2). Therefore B A is to A C, as B A is to A E (V. 11), and A C is equal to A E (V. 9). Because the angle A E C is equal to the angle A C E (I. 5). But the angle A E C is equal to the exterior and opposite angle B A D; and the angle A C E is equal to the alternate angle C A D (I. 29). Therefore the angle B A D is equal to the angle C A D (I. *Ax.* 1). Wherefore the angle B A C is bisected by the straight line A D. Therefore, if the angle, &c. Q. E. D.

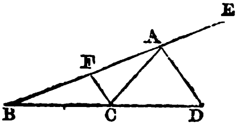
Corollary.—If the same straight line bisect an angle of a triangle and its opposite side, the triangle is isosceles.

Exercise.—*The straight line which bisects any angle of a triangle and likewise cuts the base (that is, the interior bisecting line), divides the triangle into two triangles which are to one another as their sides which contain the bisected angle.*

PROP. A. THEOREM.

If the exterior angle of a triangle be bisected by a straight line which cuts the base produced, the segments between the bisecting line and the extremities of the base, have to one another the same ratio which the adjacent sides of the triangle have; and conversely, if the segments of the base produced have the same ratio to one another which the adjacent sides of the triangle have, the straight line drawn from the vertex to the point of section, bisects the exterior angle of the triangle.

Let A B C be a triangle, having one of its sides B A produced to E; and let the outward angle C A E be bisected by the straight line A D which meets the base produced in D. The segment B D is to the segment D C, as the side B A is to the side A C.



Through the point C, draw C F parallel to A D (I. 31).

Because the straight line A C meets the parallels A D and F C, the angle A C F is equal to the alternate angle C A D (I. 29). But the angle C A D is equal to the angle D A E (*Hyp.*). Therefore the angle D A E is equal to the angle A C F (I. *Ax.* 1). Again, because the straight line F A E meets the parallels A D and F C, the exterior angle D A E is equal to the interior and opposite angle C F A (I. 29). But the angle A C F has been proved equal to the angle D A E. Therefore the angle A C F is equal to the angle C F A (I. *Ax.* 1), and the side A F to the side A C (I. 6). Because A D is parallel to F C, a side of the triangle B C F. Therefore B D is to D C, as B A to A F (VI. 2). But A F is equal to A C. Therefore B D is to D C, as B A is to A C (V. 7).

Next, let the segment B D be to the segment D C, as the side B A is to the side A C, and let A D be joined. The angle C A E is bisected by the straight line A D.

The same construction being made, because B D is to D C, as B A to A C; and B D is to D C, as B A to A F (VI. 2). Therefore B A is to A C, as B A to A F (V. 11), and A C is equal to A F (V. 9). Because the angle A F C is equal to the angle A C F (I. 5). But the angle A F C is equal to the exterior angle E A D (I. 29), and the angle A C F to the alternate angle C A D. Therefore E A D is equal to the angle C A D (I. *Ax.* 1). Wherefore the angle C A E is bisected by the straight line A D. Therefore, if the exterior, &c. Q. E. D.

This very useful proposition was added to Book VI., by Dr. Simson. It is generally considered another case of Prop. III., and it might have been incorporated with that proposition.

When the triangle A B C is isosceles, the straight line A D, which bisects the exterior angle at the vertex, is then parallel to the base, and the proposition fails. In all other cases, this exterior bisecting line cuts the base produced either on the same side with the exterior angle, or on the opposite side.

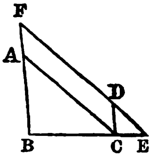
Corollary 1.—The circle described on the straight line intercepted between the points where the interior and exterior bisecting lines cut the base, passes through the vertex of the triangle.

Corollary 2.—The straight line intercepted between the point where the exterior bisecting line cuts the base produced, and the remote extremity of the base, is *harmonically* divided at the near extremity of the base, and the point where the interior bisecting line cuts the base.

PROP. IV. THEOREM.

The sides about the equal angles of equiangular triangles are proportionals; and those which are opposite to the equal angles are homologous sides, that is, are the antecedents or consequents of the ratios.

Let A B C and D C E be equiangular triangles, having the angle A B C equal to the angle D C E, and the angle A C B to the angle D E C; and consequently the angle B A C equal to the angle CDE (I. 32). The sides about the equal angles of the triangles A B C and D C E are proportionals; and the sides which are opposite to the equal angles are homologous.



Let the triangle D C E be placed, so that its side CE may be contiguous to B C, and in the same straight line with it (I. 22).

Because the angle B C A is equal to the angle C E D (*Hyp.*). To each of these equals, add the angle A B C. Therefore the two angles A B C and B C A are equal to the two angles ABC and CED (I. *Ax.* 2). But the two angles A B C and B C A are together less than two right angles (I. 17). Therefore the two angles ABC and C E D are also less than two right angles, and B A and E D if produced will meet (I. *Ax.* 12). Let them be produced and meet in the point F. Because the angle A B C is equal to the angle D C E (*Hyp.*), B F is parallel to C D (I. 28). Because the angle A C B is equal to the angle D E C, A C is parallel to F E (I. 28). Therefore F A C D is a parallelogram; and A F is equal to C D, and A C to F D (I. 34). Because A C is parallel to F E, one of the sides of the triangle F B E, B A is to A F, as B C to C E (VI. 2). But A F is equal to C D. Therefore B A is to C D, as B C is to C E (V. 7); and alternately, A B is to B C as D C is to C E (V. 16). Again, because C D is parallel to B F, B C is to C E, as F D is to D E (VI. 2). But F D is equal to A C. Therefore, B C is to CE as A C is to D E (V. 7); and alternately, B C is to C A, as C E is to E D (V. 16). Because it has been proved that AB is to B C, as D C to CE, and B C is to C A, as C E to E D. Therefore, *ex æquali*, B A is to A C as CD is to D E (V. 22). Therefore the sides, &c. Q. E. D.

This important proposition might be more easily demonstrated, by cutting from A B and A C parts equal to D C and D E, and joining the points where these parts are cut off. The triangle thus formed can be shown to be equal to the triangle D C E, and their angles are respectively equal. It can then be shown that the sides of this triangle are proportional to the sides of the triangle A B C about the common angle A. As the same construction can be made for each of the angles B and C, it is at once inferred that the sides about each of the angles of equiangular triangles are proportionals. To write out this construction and demonstration will be a useful exercise for the student.

Corollary 1.—Equiangular triangles are similar figures, and the sides opposite two equal angles are homologous.

Corollary 2.—The sides opposite two equal angles of equiangular triangles, are proportionals.

Corollary 3.—Triangles that are similar to the same triangle, are similar to one another.

Corollary 4.—Isosceles triangles which have one angle in the one equal to one angle in the other are similar.

Corollary 5.—A parallel to the base of a triangle cuts off from it a similar triangle.

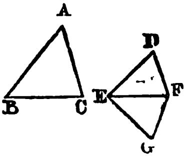
Exercise 1.—In a triangle, the straight line drawn from the vertex bisecting the base, bisects every parallel to the base intercepted by the sides.

Exercise 2.—In equiangular triangles, the perpendiculars drawn from the vertices of equal angles to the opposite sides, are proportional to those sides.

PROP. V. THEOREM.

If the sides of two triangles, about each of their angles, be proportionals, the triangles are equiangular; and the equal angles are those which are opposite to the homologous sides.

Let the triangles A B C and D E F have their sides proportionals, so that A B is to B C, as D E to E F; and B C is to C A, as E F to F D; and therefore, *ex æquali*, B A is to A C, as E D is to D F. The triangle A B C is equiangular to the triangle D E F, and the angles which are opposite to the homologous sides are equal, viz. the angle A B C equal to the angle D E F, the angle B C A to the angle E F D, and the angle B A C to the angle E D F.



At the points E and F, in the straight line E F, make the angle F E G equal to the angle A B C, and the angle E F G equal to B C A (I. 23).

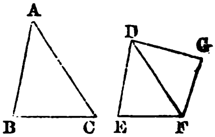
Because the remaining angle E G F, is equal to the remaining angle B A C (I. 32), the triangle G E F is equiangular to the triangle A B C. Therefore they have their sides opposite to the equal angles proportionals (VI. 4). Wherefore, A B is to B C as G E is to E F. But A B is to B C, as D E is to E F (*Hyp.*). Therefore D E is to E F, as G E is to E F (V. 11). Because D E and G E have the same ratio to E F, D E is equal to G E (V. 9). For the same reason, D F is equal to F G. Because, in the triangles D E F and G E F, D E is equal to E G, and E F is common, the two sides D E and E F are equal to the two G E and E F, each to each. But the base D F is equal to the base G F. Therefore the angle D E F is equal to the angle G E F (I. 8), and the remaining angles of the one to the remaining angles of the other, each to each. Therefore the angle D F E is equal to the angle G F E, and the angle E D F to the angle E G F. Because the angle D E F is equal to the angle G E F, and the angle G E F is equal to the angle A B C (*Const.*). Therefore the angle A B C is equal to the angle D E F (I. *Ax.* 1). For the same reason, the angle A C B is equal to the angle D F E, and the angle at A is equal to the angle at D. Therefore the triangle A B C is equiangular to the triangle D E F. Wherefore, if the sides, &c. Q. E. D.

This proposition is the converse of Prop. IV. The 4th and 5th of Book VI., and the 47th and 48th of Book I., constitute the most important principles in the Elements. They include the principles of Trigonometry and its various applications, as well as those of analytic geometry in general.

PROP. VI. THEOREM.

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals; the triangles are equiangular, and those angles are equal which are opposite to the homologous sides.

Let the triangles A B C and D E F have the angle B A C in the on equal to the angle EDF, in the other; and the sides about those angles proportionals, that is, BA to AC, as ED to DF. The triangles ABC and DEF are equiangular, and the angle ABC is equal to the angle DEF, and the angle ACB to the angle DFE.



At the points D and F, in the straight line DF, make the angle FDG equal to either of the angles BAC or EDF (I. 23); and the angle DFG equal to the angle ACB.

Because the remaining angle at B is equal to the remaining angle at G (I. 32), the triangle DGF is equiangular to the triangle ABC. Therefore BA is to AC, as GD is to DF (VI. 4). But (*Hyp.*) BA is to AC, as ED is to DF. Therefore ED is to DF, as GD is to DF (V. 11); and ED is equal to DG (V. 9). Because ED is equal to DG and DF is common to the two triangles EDF and GDF. Therefore the two sides ED and DF are equal to the two sides GD and DF, each to each. But the angle EDF is equal to the angle GDF (*Const.*). Therefore the base EF is equal to the base FG (I. 4), and the triangle EDF to the triangle GDF. Because the remaining angles of the one are equal to the remaining angles of the other, each to each. Therefore the angle DFG is equal to the angle DFE, and the angle at G to the angle at E. But the angle DFG is equal to the angle ACB (*Const.*). Therefore the angle ACB is equal to the angle DFE (I. *Ax.* 1). But the angle BAC is equal to the angle EDF (*Hyp.*). Therefore the remaining angle at B is equal to the remaining angle at E (I. 32); and the triangle ABC is equiangular to the triangle DEF. Wherefore, if two triangles, &c. Q. E. D.

Corollary 1.—**Triangles which have a common angle and the sides about it proportionals, have their bases parallel, and are equiangular and similar.**

Corollary 2.—**Triangles which have a common angle and parallel bases, are equiangular and similar.**

Exercise.—**If from any number of points in a straight line parallels be drawn proportional to the distances of these points from a given point, the straight line joining this point and the extremity of one of the parallels passes through the extremities of all the parallels.**

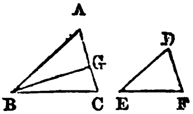
PROP VII. THEOREM.

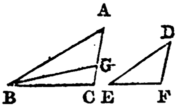
If two triangles have one angle of the one equal to one angle of the other, and the sides about another angle in each, proportionals; and if the remaining angle in each be of the same affection (that is, either both acute, or both not acute); the two triangles are equiangular and similar.

Let the two triangles ABC and DEF have one angle BAC in the one equal to the angle EDF in the other, and the sides about their two other angles ABC and DEF, proportionals, so that AB is to BC as DE is to EF; and let their remaining angles ACB and DFE be of the same affection, that is, either both acute, or both not acute. The triangles ABC and DEF are equiangular and similar.

For if the angle ABC is not equal to the angle DEF one of them *must be greater than* the other. Let ABC be the greater angle, *and at the point B in* the straight line AB, make the angle ABG *equal to the angle DEF* (I. 23).

Because in the two triangles ABG and DEF, the angle at A is equal to the angle at D (*Hyp.*) and the angle ABG is equal to the angle DEF (*Const.*). Therefore the remaining angle AGB is equal to the remaining angle DFE (I. 32), and the triangle ABG is equiangular to DEF. Because these triangles are equiangular AB is to BG, as DE is to EF (VI. 4). But AB is to BC as DE is to EF (*Hyp.*). Therefore AB is to BC, as AB is to BG (V. 11); and BG is equal to BC (V. 9). Wherefore the triangle GBC is isosceles, and the angle BGC is equal to the angle BCG. Therefore the angles BGC and BGA are both acute, or both not acute, according as the angles ACB and DFE, are both acute or both not acute. If they are both acute, as in the first figure, the angles BGC and BGA which BG makes with AC are together less than two right angles, which is impossible (I. 13). If they are both not acute, as in the second figure, the angles BGC and BCG of the triangle BGC, are together not less than two right angles, which is also impossible (I. 17). Therefore the angle ABC is not unequal to the angle DEF; that is, the angle ABC is equal to the angle DEF. Wherefore the remaining angle ACB is equal to the remaining angle DFE; and the triangle ABC is equiangular and similar to the triangle DEF. Q. E. D.





In this proposition, we have considerably departed from Euclid, in order to make the demonstration more plain and easy to the learner. Besides, Euclid has three separate cases to demonstrate, which are here reduced to two, and the same proof is made, with a very slight exception, to apply to both.

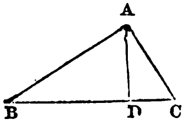
Corollary.—If two triangles have two sides in the one proportional to two sides in the other, and the angles opposite one pair of the homologous sides equal, the angles opposite the other pair of homologous sides are either equal or supplementary.

PROP. VIII. THEOREM.

In a right-angled triangle, if a perpendicular be drawn from the right angle to the opposite side, the triangles on each side of the perpendicular are similar to the whole triangle, and to one another.

Let ABC be a right-angled triangle, having the right angle BAC, and from the point A let AD be drawn perpendicular to the opposite side BC. The triangles ABD and ADC are similar to the whole triangle ABC, and to one another.

Because the angle BAC is equal to the angle ADB, each of them being a right angle (I. *Ax.* 11), and the angle at B is common to the two triangles ABC and ABD. Therefore the remaining angle ACB is equal (I. 32) to the remaining angle BAD; and the triangle ABC is equiangular to the triangle ABD. But the sides about their equal angles are proportionals (VI. 4). Therefore the triangles are similar (VI. *Def.* 1). In like manner it may be shown that the triangle ADC is equiangular and similar to the *triangle ABC.* Because the triangles ABD and ACD, are both



equiangular and similar to A B C. Therefore they are equiangular and similar to each other. Therefore, in a right-angled, &c. Q. E. D.

COROLLARY.—From this it is manifest, that the perpendicular drawn from the right angle of a right-angled triangle to the hypotenuse, is a mean proportional between the segments of the hypotenuse; and also that each of the sides is a mean proportional between the hypotenuse, and the segment of it adjacent to that side. For in the triangles B D A and A D C, B D is to D A, as D A is to D C. In the triangles A B C and D B A, B C is to B A, as B A is to B D; and in the triangles A B C and A C D, B C is to C A, as C A is to C D (VI. 4).

The preceding corollary to this proposition is the most important part of it, as will be seen by its subsequent application. From this corollary, in combination with Props. XVI. and XVII. of this Book, and Prop. II. Book II., a new and independent demonstration of Prop. XLVII. Book I., may be obtained, which is considerably shorter than that given by Euclid.

Corollary 1.—The hypotenuse is to either leg of the right-angled triangle, as the other leg is to the perpendicular drawn to the hypotenuse from the right angle.

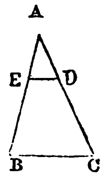
Corollary 2.—In a triangle, if the triangles formed by the perpendicular and the sides be similar, the angles which it makes with the sides must be either equal or complementary. If the angles be equal, the perpendicular bisects the vertical angle, the similar triangles are equal, and the whole triangle is isosceles. If these angles be not equal, and the perpendicular be within the triangle, they must be equal to the alternate angles at the base, complementary to each other, and the vertical angle being then equal to the sum of the angles at the base, is a right angle. If the perpendicular be without the triangle, the angles between it and the sides are complementary, and the vertical angle being equal to the difference between these angles, is an acute angle.

Corollary.—If the base, the two sides, and the perpendicular of a triangle be proportionals, it is a right-angled triangle.

PROP. IX. PROBLEM.

From a given straight line to cut off any required part, or submultiple.

Let A B be the given straight line. It is required to cut off any part or submultiple from it.



From the point A draw a straight line A C, making any angle B A C, with A B. In A C take any point D, and make A C the same multiple of A D, that A B is of the part to be cut off from it. Join B C, and draw D E parallel to C B. The part A E is the part required to be cut off from A B.

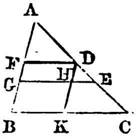
Because E D is parallel to B C, one of the sides of the triangle A B C, C D is to D A, as B E is to E A (VI. 2). Therefore by composition, C A is to A D, as B A is to A E (V. 18). But C A is a multiple of A D (*Const.*). Therefore B A is the same multiple of A E (V. D); and whatever part A D is of A C, A E is the same part of A B. Wherefore, from the straight line A B is cut off the part or submultiple required. Q. E. F.

Corollary.—By drawing through the points in A C, which mark off the successive parts of it each equal to A D, parallels to the straight line B C the line A B will be divided into the same number of equal parts. Thus a given straight line may be divided into any number of equal parts required.

PROP. X. PROBLEM.

To divide a given straight line similarly to a given divided straight line, that is, into parts proportional to the parts of the given divided straight line.

Let AB be the given straight line to be divided, and AC the given straight line, divided into parts at D and E. It is required to divide AB into parts proportional to the parts of AC.



Place AB and AC so as to make any angle BAC with each other; and join BC. Through the points D and E draw DF and EG parallels to BC (I. 31). The straight line AB is divided, at the points F and G, into parts proportional to those of AC. Through D draw DHK parallel to AB (I. 31).

Because FH and HB are parallelograms (*Const.*). Therefore DH is equal to FG, and HK to GB (I. 34). Because HE is parallel to KC, one of the sides of the triangle DKC, CE is to ED, as KH is to HD (VI. 2). But BG is equal to KH, and GF to HD (*Const.*). Therefore CE is to ED, as BG is to GF (V. 7). Again, because FD is parallel to GE, one of the sides of the triangle AGE, ED is to DA, as GF is to FA (VI. 2). Therefore, as has been proved, CE is to ED, as BG is to GF, and ED is to DA, as GF is to FA. Wherefore the given straight line AB is divided into parts proportional to the parts of AC. Q. E. F.

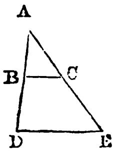
Exercise 1.—To divide a given straight line, into two parts that shall have to each other a given ratio.

Exercise 2.—To produce a given straight line, so that the whole line thus produced may have to the part produced a given ratio.

PROP. XI. PROBLEM.

To find a third proportional to two given straight lines.

Let AB and AC be the two given straight lines. It is required to find a third proportional to AB and AC.



Place AB and AC so as to make any angle BAC with each other. Produce AB and AC to the points D and E; making BD equal to AC (I. 3). Join BC, and through D, draw DE parallel to BC (I. 31). The straight line CE is a third proportional to AB and AC.

Because BC is parallel to DE, a side of the triangle ADE, AB is to BD, as AC is to CE (VI. 2). But BD is equal to AC. Therefore AB is to AC, as AC is to CE (V. 7). Wherefore, to the two given straight lines AB and AC, a third proportional CE is found. Q. E. F.

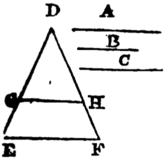
Exercise.—If in the preceding figure, instead of making BD equal to AC, AD had been made equal to AC, and DE drawn parallel to BC, then AE would have been the third proportional. Required the demonstration.

This problem may be solved in a variety of ways, especially by the application of *Prop. VIII.*, and its corollary. It will be useful exercise for the student, to find *out some of these solutions.* In the same way, the problem may be extended to *the mode of finding* a series of continual proportionals.

PROP. XII. PROBLEM.

To find a fourth proportional to three given straight lines.

Let A, B, and C be the three given straight lines. It is required to find a fourth proportional to A, B, and C.



Take two straight lines DE and DF, containing any any angle EDF. Upon these straight lines make DG equal to A, GE equal to B, and DH equal to C (I. 3). Join GH, and through E draw EF parallel to GH (I. 31). The straight line HF is the fourth proportional to A, B, and C.

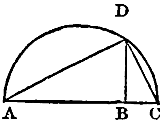
Because GH is parallel to EF, one of the sides of the triangle DEF, DG is to GE, as DH is to HF (VI. 2). But DG is equal to A, GE to B, and DH to C. Therefore A is to B, as C is to HF (V. 7). Wherefore, to the three given straight lines A, B, and C, a fourth proportional HF is found. Q. E. F.

This problem may also be solved in a variety of ways, chiefly by the application of Props. XXXV. and XXXVI. Book III., and Prop. VIII. of this Book. The ingenious student will find it a useful exercise to try and find out these solutions.

PROP. XIII. PROBLEM.

To find a mean proportional between two given straight lines.

Let AB, and BC be the two given straight lines. It is required to find a mean proportional between them.



Place AB and BC in a straight line adjacent to each other; and upon AC describe the semi-circle ADC. From the point B, draw BD at right angles to AC (I. 11). The straight line BD is a mean proportional between AB and BC. Join AD and DC.

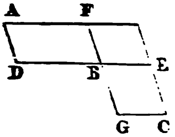
Because the angle ADC in a semicircle is a right (III. 31), and BD is drawn from the right angle perpendicular to the opposite side AC of the triangle ADC. Therefore DB is a mean proportional between AB and BC, the segments of the base (VI. 8, *Cor.*). Wherefore between the two given straight lines AB and BC, a mean proportional DB is found. Q. E. F.

Other modes of solving this problem are suggested to the student by the consideration of Prop. VIII. of this Book, and Props. XXXI. and XXXVI. of Book III. To find *two* mean proportionals between two given straight lines, is beyond the power of elementary geometry. This problem was connected with the famous problem of the "*duplication* (doubling) *of the cube*;" for, if four straight lines be continually proportional, the first is to the fourth, in the *triplicate ratio* of the first to the second; or, as the *cube* of the first is to the *cube* of the second.

PROP. XIV. THEOREM.

Equal parallelograms, which have an angle of the one equal to an angle of the other, have their sides about the equal angles reciprocally proportional; and conversely, parallelograms which have an angle of the one equal to an angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

First, Let *AB* and *BC* be equal parallelograms, which have their angles at B equal. The sides of the parallelograms A B and B C about the equal angles, are reciprocally proportional; that is, D B is to B E, as G B is to B F.



Piace the parallelograms so that the sides D B and B E shall be in the same straight line; F B and B G shall be in one straight line (I. 14). Complete the parallelogram F E.

Because the parallelogram A B is equal to the parallelogram B C, and F E is another parallelogram. Therefore A B is to F E, as B C is to F E (V. 7). But A B is to F E, as the base D B is to the base B E (VI. 1); and B C is to F E, as the base G B is to the base B F. Therefore D B is to B E, as G B is to B F (V. 11). Wherefore the sides of the parallelograms A B and B C about their equal angles are reciprocally proportional.

Next, let the sides about the equal angles of the parallelograms A B and B C be reciprocally proportional,—viz., D B to B E, as G B to B F. The parallelogram A B is equal to the parallelogram B C.

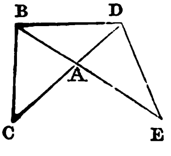
Because, D B is to B E, as G B is to B F. But D B is to B E, as the parallelogram A B is to the parallelogram F E (VI. 1); and G B is to B F as the parallelogram B C is to the parallelogram F E. Therefore A B is to F E, as B C is to F E (V. 11). Wherefore the parallelogram A B is equal to the parallelogram B C (V. 9). Therefore equal parallelograms, &c. Q. E. D.

Corollary.—If two parallelograms are equal, and have their sides reciprocally proportional, they are equiangular.

PROP. XV. THEOREM.

Equal triangles which have an angle of the one equal to an angle of the other, have their sides about the equal angles reciprocally proportional; and conversely, triangles which have an angle in the one equa to an ang e in the other, and their sides about the equal angles reciprocally proporticnal, are equal to one another.

First, let A B C and A D E be equal triangles, which have the angle B A C equal to the angle D A E. The sides about the equal angles of the triangles are reciprocally proportional; that is, C A is to A D, as E A is to A B.



Place the triangles so that their sides C A and A D shall be in one straight line; E A and A B shall be in one straight line (I. 14). Join B D.

Because the triangle A B C is equal to the triangle A D E, and A B D is another triangle. Therefore the triangle C A B, is to the triangle B A D, as the triangle A E D is to the triangle D A B (V. 7). But the triangle C A B is to the triangle B A D, as the base C A is to the base A D (VI. 1); and the triangle E A D is to the triangle D A B as the base E A is to the base A B (VI. 1). Therefore C A is to A D, as E A is to A B (V. 11). Wherefore the sides of the triangles A B C, A D E, about the equal angles are reciprocally proportional.

Next, let the sides of the triangles A B C and A D E about their equal angles *at A be* reciprocally proportional,—viz., C A to A D, as E A to *A B. The* triangle A B C is equal to the triangle A D E. Join B D.

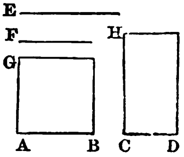
Because C A is to A D, as E A is to A B (*Hyp.*). But C A is to A D, as the triangle A B C is to the triangle B A D (VI. 1); and E A is to A B, as the triangle E A D is to the triangle B A D. Therefore the triangle B A C is to the triangle B A D, as the triangle E A D is to the triangle B A D (V. 11). Wherefore the triangle A B C is equal to the triangle A D E (V. 9). Therefore, equal triangles, &c. Q. E. D.

Corollary.—Equal triangles which have an angle in the one supplementary to an angle in the other, have their sides about the supplementary angles reciprocally proportional; and conversely, triangles which have an angle in the one supplementary to an angle in the other, and their sides about the supplementary angles reciprocally proportional, are equal to one another.

PROP. XVI. THEOREM.

If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means; and conversely, if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportionals.

First, let the four straight lines AB, CD, E and F be proportionals, viz., AB to CD, as E to F. The rectangle contained by A B and F, is equal to the rectangle contained by C D and E.



From the points A and C draw A G and C H at right angles to A B and C D (I. 11). Make A G equal to F, and C H equal to E (I. 3). Complete the parallelograms B G and D H (I. 31).

Because AB is to CD, as E is to F, and E is equal to CH, and F to A G. Therefore A B is to C D as C H is to A G (V. 7); and the sides of the parallelograms B G and D H about the equal angles are reciprocally proportional. Therefore the parallelogram B G is equal to the parallelogram D H (VI. 14). But the parallelogram B G is contained by the straight lines A B and F; because A G is equal to F. And the parallelogram D H is contained by the straight lines CD and E; because C H is equal to E. Therefore the rectangle contained by the straight lines A B and F, is equal to the rectangle contained by the straight lines C D and E.

Next, if the rectangle contained by the straight lines A B and F, be equal to the rectangle contained by the straight lines C D and E; the four straight lines are proportionals, that is, A B is to CD, as E is to F.

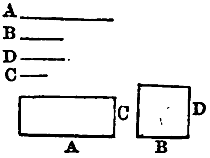
The same construction being made, the rectangle contained by the straight lines AB and F, is equal to the rectangle contained by C D and E (*Hyp.*). But the rectangle B G is contained by A B and F; because A G is equal to F. And the rectangle D H is contained by the straight lines C D and E; because C H is equal to E. Therefore the parallelogram B G is equal to the parallelogram D H (I. *Ax.* 1); and they are equiangular. But the sides about the equal angles of equal parallelograms are reciprocally proportional (VI. 14). Therefore, A B is to C D, as C H is to A G. But C H is equal to E, and A G to F. Therefore AB is to CD, as E is to F (V. 7). Wherefore if four, &c. Q. E. D.

Corollary.—*Equal triangles* and parallelograms have their bases and altitudes *reciprocally proportional*; and *conversely*, triangles and parallelograms which have *their bases and* altitudes reciprocally proportional, are equal.

PROP. XVII. THEOREM.

If three straight lines be proportionals, the rectangle contained by the extremes is equal to the square of the mean; and conversely, if the rectangle contained by the extremes be equal to the square of the mean, the three straight lines are proportionals.

First, let the three straight lines A, B, and C be proportionals, viz., A to B, as B to C. The rectangle contained by A and C is equal to the square of B.



Take the straight line D equal to the straight line B.

Because A is to B, as B is to C, and B is equal to D. Therefore A is to B, as D is to C (V. 7). But when four straight lines are proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means (VI. 16). Therefore the rectangle contained by A and C is equal to the rectangle contained by B and D. But the rectangle contained by B and D, is the square of B, because B is is equal to D. Therefore the rectangle contained by A and C, is equal to the square of B.

Next, if the rectangle contained by A and C, be equal to the square of B, the three straight lines are proportionals, that is, A is to B, as B is to C.

The same construction being made, the rectangle contained by A and C is equal to the square of B (*Hyp.*). But the square of B is equal to the rectangle contained by B and D, because B is equal to D (*Const.*). Therefore the rectangle contained by A and C, is equal to that contained by B and D (I. *Ax.* 1). But if the rectangle contained by the extremes be equal to that contained by the means, the four straight lines are proportionals (VI. 16). Therefore A is to B, as D is to C. But B is equal to D. Therefore A is to B, as B is to C. Wherefore, if three straight lines, &c. Q. E. D.

Exercise 1.—To construct a square equal to a parallelogram, whose base and altitude are given.

Exercise 2.—To construct a square equal to a triangle, whose base and altitude are given.

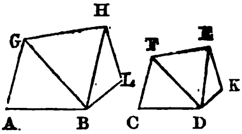
Exercise 3.—Demonstrate the 47th proposition of the first Book, in the manner suggested in the annotation to Prop. VIII.

DEFINITION.—Similar figures are said to be similarly situated, when their homologous sides are parallel and drawn in the same direction. This definition is given to explain the enunciation of the next proposition.

PROP. XVIII. PROBLEM.

Upon a given straight line, to describe a rectilineal figure similar, and similarly situated, to a given rectilineal figure.

First let AB be the given straight line, and CDEF the given rectilineal *figure of four* sides. It is required to describe upon the given straight line AB a rectilineal figure similar, and similarly situated, to the rectilineal figure CDEF.



Join DF, and at the points A and B in the straight line AB, make the angle BAG equal to the angle DCF (I. 23), and the angle ABG equal to the angle CDF. Because the remaining angle AGB is equal to the remaining angle CFD (I. 32, and *Ax.* 3). Therefore the triangle GAB is equiangular to the triangle FCD. Again, at the points G and B, in the straight line GB, make the angle BGH equal to the angle DFE (I. 23), and the angle GBH equal to the angle FDE. Because the remaining angle GHB is equal to the remaining angle FED. Therefore the triangle GBH is equiangular to the triangle FDE. Because the angle AGB is equal to the angle CFD, and the angle BGH to the angle DFE. Therefore the whole angle AGH is equal to the whole angle CFE (I. *Ax.* 2). For the same reason, the whole angle ABH is equal to the whole angle CDE. Also the angle BAG is equal to the angle DCF (*Const.*), and the angle GHB to the angle FED. Therefore the rectilineal figure ABHG is equiangular to the rectilineal figure CDEF. Likewise these figures have their sides about the equal angles proportionals. Because the triangles GAB and FCD are equiangular, BA is to AG, as DC is to CF (VI. 4), and AG is to GB, as CF is to FD. Because the triangles BGH and DFE are equiangular, therefore GB is to GH, as FD is to FE. Wherefore, *ex æquali*, AG is to GH as CF is to FE (V. 22). In the same manner, it may be proved that AB is to BH, as CD is to DE · and GH is to HB, as FE is to ED (VI. 4). Because the rectilineal figures ABHG and CDEF are equiangular, and have their sides about the equal angles proportionals. Therefore they are similar to one another (VI. *Def.* 1).

Next, let it be required to describe upon the given straight line AB, a rectilineal figure similar, and similarly situated, to the rectilineal figure CDKEF of five sides.

Join DE, and upon the given straight line AB describe the rectilineal figure ABHG similar, and similarly situated, to the quadrilateral figure CDEF, by the former case. At the points B and H, in the straight line BH, make the angle HBL equal to the angle EDK, and the angle BHL equal to the angle DEK (I. 23).

Because the two angles HBL and BHL are equal to the two angles EDK and DEK. Therefore the remaining angle BLH is equal to the remaining angle DKE (I. 32). Because the figures ABHG and CDEF are similar, the angle GHB is equal to the angle FED (VI. *Def.* 1), but the angle BHL is equal to the angle DEK. Therefore the whole angle GHL is equal to the whole angle FEK. For the same reason the whole angle ABL is equal to the whole angle CDK. Wherefore the five-sided figures AGHLB and CFEKD are equiangular. Because the figures AGHB and CFED are similar. Therefore GH is to HB, as FE is to ED (VI. *Def.* 1). But HB is to HL as ED is to EK (*VI. 4*). Therefore, *ex æquali*, GH is to HL as FE is to EK (V. 22). *For the same reason*, AB is to BL, as CD is to DK. Because the *triangles BLH* and *DKE* are equiangular. Therefore BL is to LH, as

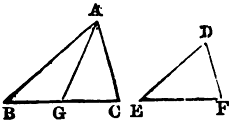
D K is to K E (VI. 4). Because the five-sided figures A G H L B and C F E K D are equiangular, and have their sides about the equal angles proportionals. Therefore they are similar to one another. In the same manner, a rectilineal figure may be described upon a given straight line similar to a given rectilineal figure of six sides; and so on. Q. E. F.

This problem is more easily effected in practice, by placing the given straight line A B in the same straight line with, or parallel to, the assumed base C D, and drawing straight lines parallel to the sides of the figure, and of the triangles into which it is divided by the straight lines drawn from the point D to its different angles. The proof is thus also more easily established.

PROP. XIX. THEOREM.

Similar triangles are to one another in the duplicate ratio of their homologous sides.

Let A B C and D E F be similar triangles, and let the angle A B C be equal to the angle D E F, and let A B be to B C, as D E to E F, so that the side B C is homologous to E F (V. *Def.* 12). The triangle A B C has to the triangle D E F, the duplicate ratio of that which the side B C has to the side E F.



Take B G a third proportional to B C and E F (VI. 11); so that B C is to E F, as E F to B G. Join G A.

Because A B is to B C, as D E is to E F (*Hyp.*). Therefore, alternately, A B is to D E, as B C is to E F (V. 16). But B C is to E F, as E F is to B G (*Const.*). Therefore A B is to D E, as E F is to B G (V. 11). Wherefore the sides of the two triangles A B G and D E F, which are about the equal angles A B G and D E F, are reciprocally proportional. But triangles, which have the sides about two equal angles reciprocally proportional, are equal to one another (VI. 15). Therefore the triangle A B G is equal to the triangle D E F. Because B C is to E F, as E F is to B G; and if three straight lines be proportionals, the first is said to have to the third, the duplicate ratio of that which it has to the second (V. *Def.* 10). Therefore B C has to B G the duplicate ratio of that which B C has to E F. But B C is to B G, as the triangle A B C is to the triangle A B G (VI. 1). Therefore the triangle A B C has to the triangle A B G, the duplicate ratio of that which B C has to E F. But the triangle A B G has been proved equal to the triangle D E F. Therefore also the triangle A B C has to the triangle D E F, the duplicate ratio of that which the side B C has to the side E F. Therefore similar triangles, &c. Q. E. D.

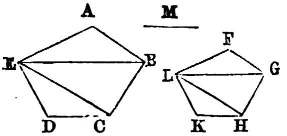
COROLLARY.—From this it is manifest, that if three straight lines be proportionals, as the first is to the third, so is any triangle upon the first to a similar and similarly described triangle upon the second.

This proposition is one of the most important in the Elements, and as such ought to be carefully studied and remembered by the student. It is the basis on which the comparison of the areas of similar rectilinear figures is founded: and it thus furnishes the rule by which, when the area of one rectilineal *figure or polygon* is known, the areas of all similar rectilineal figures or polygons are obtained.

PROP. XX. THEOREM.

Similar polygons are divisible into the same number of similar triangles; these triangles have to one another the same ratio that the polygons have; and the polygons have to one another the duplicate ratio of that which their homologous sides have.

Let ABCDE and FGHKL be similar polygons, and let AB be the side homologous to FG. The polygons ABCDE and FGHKL are divisible into the same number of similar triangles. These triangles have to one another the same ratio which the polygons have. And the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the side AB has to the side FG.



Join BE, EC, GL, and LH.

Because the polygon ABCDE is similar to the polygon FGHKL, the angle BAE is equal to the angle GFL, and BA is to AE, as GF is to FL (VI. *Def.* 1). Because the triangles ABE and FGL have an angle in the one, equal to an angle in the other, and their sides about the equal angles proportionals. Therefore the triangle ABE is equiangular and similar to the triangle FGL (VI. 6 and 4). Wherefore the angle ABE is equal to the angle FGL. Because the polygons are similar, the whole angle ABC is equal to the whole angle FGH (VI. *Def.* 1). Therefore the remaining angle EBC is equal to the remaining angle LGH (I. 32, and *Ax.* 3). Because the triangles ABE and FGL are similar, EB is to BA, as LG is to GF (VI. 4). Because the polygons are similar, AB is to BC, as FG is to GH (VI. *Def.* 1). Therefore, *ex æquali*, EB is to BC, as LG is to GH (V. 22). Because the sides about the equal angles EBC and LGH are proportionals. Therefore the triangle EBC is equiangular, and similar to the triangle LGH (VI. 6 and 4). For the same reason, the triangle ECD is similar to the triangle LHK. Therefore the similar polygons ABCDE and FGHKL are divided into the same number of similar triangles.

Again, because the triangles are similar, the triangle ABE has to the triangle FGL the duplicate ratio of that which the side BE has to the side GL (VI. 19). For the same reason, the triangle BEC has to the triangle GLH the duplicate ratio of that which BE has to GL. Therefore the triangle ABE is to the triangle FGL, as the triangle BEC is to the triangle GLH (V. 11). Because the triangles are similar, the triangle EBC has to the triangle LGH the duplicate ratio of that which the side EC has to the side LH. For the same reason, the triangle ECD has to the triangle LHK the duplicate ratio of that which EC has to LH. Therefore the triangle EBC is to the triangle LGH, as the triangle ECD is to the triangle LHK (V. 11). But it has been proved, that the triangle EBC is to the triangle LGH, as the triangle ABE is to the triangle FGL. Therefore the triangle ABE is to the triangle FGL, as the triangle EBC is to the triangle LGH, and as the triangle ECD is to the triangle LHK. But one of the antecedents is *to one of the* consequents, as all the antecedents are to all the consequents *(V. 12)*. Therefore the triangle ABE is to the triangle FGL, as the polygon *ABCDE* is to the polygon FGHKL. But the triangle

A B E has to the triangle F G L the duplicate ratio of that which the side A B has to the homologous side F G (VI. 19). Therefore also the polygon A B C D E has to the polygon F G H K L the duplicate ratio of that which A B has to the homologous side F G. Wherefore, similar polygons, &c. Q. E. D.

Corollary 1.—In like manner it may be proved, that similar four-sided figures, or figures of any number of sides, are, one to another in the duplicate ratio of their homologous sides; and this has been proved of triangles (VI. 19). Therefore, universally, similar rectilineal figures are to one another in the duplicate ratio of their homologous sides.

Corollary 2.—If to A B and F G, two of the homologous sides of the polygon, a third proportional M be taken (VI. 11), A B has to M the duplicate ratio of that which AB has to FG (V. *Def.* 10). But the four-sided figure or polygon upon AB, has to the four-sided figure or polygon upon F G likewise the duplicate ratio of that which A B has to F G (VI. 20, *Cor.* 1). Therefore AB is to M, as the figure upon AB to the figure upon F G (V. 11); and this has been proved in triangles (VI. 19, *Cor.*). Therefore, universally, it is manifest, that if three straight lines be proportionals, the first is to the third, as any rectilineal figure upon the first is to a similar and similarly described rectilineal figure upon the second.

Corollary 3.—Because all squares are similar figures, the ratio of any two squares to one another, is the same as the duplicate ratio of their sides, for all their sides are homologous. Therefore two similar triangles are to one another as the squares of their homologous sides (V. 11). Therefore, also generally, any two similar rectilineal figures or polygons are to one another as the squares of their homologous sides.

Corollary 4.—The *perimeters* (that is, the sums of all their sides) of similar rectilineal figures or polygons are to one another as their homologous sides, homologous diagonals, or homologous altitudes.

Corollary 5.—Similar rectilineal figures or polygons are to one another as the squares of homologous diagonals, or homologous altitudes; or, as the squares of their perimeters.

Corollary 6.—Rectilineal figures or polygons are similar, which are divisible into the same number of similar and similarly situated triangles.

Corollary 7.—If similar rectilineal figures similarly described upon straight lines be equal, these straight lines are also equal.

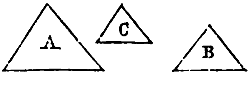
Exercise.—To construct a rectilineal figure similar to a given rectilineal figure and having a given ratio to it.

PROP. XXI THEOREM.

Rectilineal figures which are similar to the same rectilineal figure, are similar to one another.

Let each of the rectilineal figures A and B be similar to the rectilineal figure C. The figure A is similar to the figure B.

Because the figure A is similar to the figure C, they are equiangular, nd have their sides about the equal angles roportional (VI. *Def.* 1). Because the gure B is similar to the figure C, they are quiangular, and have their sides about the qual angles proportionals (VI. *Def.* 1).



Therefore *the figures* A and B are each of them equiangular to the gure *C, and* have the sides about their equal angles proportionals.

Wherefore the rectilineal figures A and B are equiangular (I. *Ax.* 1), and have their sides about the equal angles proportionals (V. 11). Therefore the figure A is similar to the figure B (VI. *Def.* 1). Therefore, rectilineal figures, &c. Q. E. D.

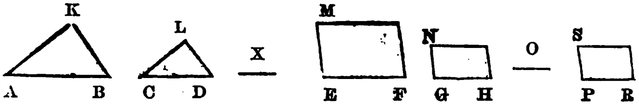
As it is inferred at once from Prop. IV. of this Book, that equiangular triangles are similar; and that triangles which are similar to the same triangle are similar to one another; it is plain that this proposition refers entirely to the case of similar quadrilateral figures, and polygons or rectilinear figures of more sides than four. The figures, therefore, instead of being triangles, should have been polygons.

PROP. XXII. THEOREM.

If four straight lines be proportionals, the similar rectilineal figures similarly described upon them, are proportionals; and conversely, if the similar rectilineal figures similarly described upon four straight lines be proportionals, these straight lines are proportionals.

First, let the four straight lines AB, CD, EF, and GH, be proportionals, that is, AB to CD, as EF to GH; upon AB and CD let the similar rectilineal figures KAB and LCD be similarly described; and upon EF and GH, the similar rectilineal figures MF and NH similarly described. The rectilineal figures KAB, LCD, MF, and NH are also proportionals, that is, KAB is to LCD, as MF is to NH.

To the straight lines AB and CD, find a third proportional X



(VI. 11); and to the straight lines EF and GH a third proportional O.

Because AB is to CD, as EF to GH, and CD is to X as GH is to O (V. 11). Therefore, *ex æquali*, AB is to X as EF is to O (V. 22). But AB is to X, as the rectilineal figure KAB is to the rectilineal figure LCD (VI. 20, *Cor.* 2); and EF is to O, as the rectilineal figure MF is to the rectilineal figure NH. Therefore, the rectilineal figure KAB is to the rectilineal figure LCD, as the rectilineal figure MF is to the rectilineal figure NH (V. 11).

Secondly, let the rectilineal figures K A B, LCD, MF, and NH be proportionals. The straight lines AB, CD, EF, and GH are also proportionals.

To the straight lines AB, CD, and EF, find a fourth proportional PR; that is, make as AB is to CD, so EF to PR (VI. 12). Upon PR describe the rectilineal figure SR, similar and similarly situated to either of the figures MF or NH (VI. 18).

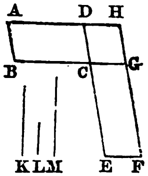
Because AB is to CD, as EF is to PR (*Const.*), and on AB and CD are described similar and similarly situated rectilineal figures KAB and LCD (*Hyp.*), and on EF and PR the similar and similarly situated rectilineal figures MF and SR (*Const.*). Therefore the rectilineal figures KAB, LCD, MF, and SR, are proportionals; that is, KAB *is to LCD, as MF* is to SR. But KAB is to LCD as MF is to NH *(Hyp.). Therefore,* MF has the same ratio to each of the rectilineal *figures NH and SR.* Wherefore they are equal to each other (V. 9), and they are similar and similarly situated (*Const.*). Therefore the straight line G H is equal to the straight line P R (VI. 20, *Cor.* 7). Because A B is to C D, as E F is to P R, and P R is equal to G H. Therefore A B is to C D, as E F is to G H (V. 7). Q. E. D.

Corollary.—If any number of straight lines be continual proportionals, the similar rectilineal figures similarly described upon them, are also continual proportionals; and *conversely*.

PROP. XXIII. THEOREM.

iangular parallelograms have to one another the ratio which is compounded of the ratios of their sides

Let A C and C F be equiangular parallelograms, having the angle B C D equal to the angle E C G. The ratio of the parallelogram A C to the parallelogram C F, is the same with the ratio which is compounded of the ratios of their sides.



Place B C and C G in a straight line; and D C and C E shall also be in a straight line (I. 14). Complete the parallelogram D G (I. 31). Take any straight line K, and find L a fourth proportional to B C, C G, and K (VI. 12); and M a fourth proportional to D C, C E, and L.

Because the ratios of K to L, and L to M, are the same with the ratios of the sides; that is, of B C to C G, and D C to C E (*Const.*): but the ratio of K to M is that which is said to be compounded of the ratios of K to L, and L to M (V. *Def.* A). Therefore K has to M the ratio compounded of the ratios of the sides. Because B C is to C G, as the parallelogram A C is to the parallelogram C H (VI. 1). But B C is to C G, as K is to L (*Const.*). Therefore K is to L, as the parallelogram A C is to the parallelogram C H (V. 11). Again, because D C is to C E, as the parallelogram C H is to the parallelogram C F (VI. 1). But D C is to C E, as L is to M (*Const.*). Therefore L is to M, as the parallelogram C H is to the parallelogram C F (V. 11). Wherefore, *ex æquali*, K is to M, as the parallelogram A C is to the parallelogram C F (V. 22). But K has to M the ratio which is compounded of the ratios of the sides. Therefore also the parallelogram A C has to the parallelogram C F, the ratio which is compounded of the ratios of the sides. Wherefore, equiangular parallelograms, &c. Q. E. D.

Otherwise.—Because there are three parallelograms A C, C H, and C F. The first A C has to the third C F the ratio which is compounded of the ratio of the first A C to the second C H, and of the ratio of the second C H to the third C F (V. *Def.* A). But A C is to C H, as B C is to C G; and C H is to C F, as C D is to C E (VI. 1). Therefore A C has to C F the ratio which is compounded of the ratios which are the same with the ratios of the sides.

Corollary 1.—Triangles which have an angle of the one equal to an angle of the other, are to one another as the rectangles contained by the sides about those angles.

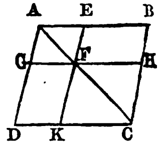
Corollary 2.—Equiangular parallelograms are to one another as the rectangles contained by their adjacent sides.

Corollary 3.—Equiangular triangles and parallelograms are to one another as the *rectangles contained* by their bases and altitudes.

PROP. XXIV. THEOREM.

Parallelograms about the diagonal of any parallelogram, are similar to the whole, and to one another.

Let B D be a parallelogram, of which the diameter is A C; and E G and H K parallelograms about the diagonal. The parallelograms E G and H K are similar to the whole parallelogram B D, and to each other.



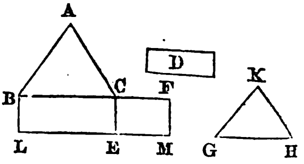
Because D C and G F are parallels, the angle A D C is equal to the angle A G F (I. 29). Because B C and E F are parallels, the angle A B C is equal to the angle A E F. Because each of the angles B C D and E F G is equal to the opposite angle D A B (I. 34). Therefore they are equal to one another. Wherefore the parallelograms B D and E G, are equiangular. Because the angle A B C is equal to the angle A E F, and the angle B A C common to the two triangles B A C and E A F, they are equiangular to one another. Therefore A B is to B C, as A E is to E F (VI. 4). Because the opposite sides of parallelograms are equal to one another (I. 34). Therefore A B is to A D, as A E is to A G (V. 7); D C to C B, as G F to F E; and C D to D A, as F G to G A. Therefore the sides of the parallelograms B D and E G about the equal angles are proportionals; and they are similar to one another (VI. *Def.* 1). For the same reason, the parallelogram B D is similar to the parallelogram H K. Because each of the parallelograms G E and K H is similar to the parallelogram D B; and rectilineal figures which are similar to the same rectilineal figure are similar to one another (VI. 21). Therefore the parallelogram G E is similar to the parallelogram K H. Wherefore, parallelograms, &c. Q. E. D.

Exercise.—**Prove from this proposition and the XIVth of this Book, that the complements of the parallelograms about the diagonal of any parallelogram are equal (I. 43).**

PROP. XXV. PROBLEM.

To describe a rectilineal figure similar to one given rectilineal figure, and equal to another given rectilineal figure.

Let A B C be one given rectilineal figure, and D another given rectilineal figure. It is required to describe a rectilineal figure similar to the rectilineal figure A B C, and equal to the rectilineal figure D.



Upon the straight line B C, describe the parallelogram B E equal to the figure A B C (I. 45. *Cor.*). To the straight line C E, apply the parallelogram C M equal to D (I. 45. *Cor.*), and having the angle F C E equal to the angle C B L. Because these angles are equal, B C and C F are in a straight line, as also L E and E M (I. 29. and I. 14). Between B C and C F find a mean proportional G H

(VI. 13). Upon G H describe the rectilineal figure K G H similar and similarly situated to the figure A B C (VI. 18).

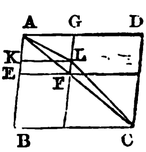
Because B C is to G H, as G H is C F; and if three straight lines be proportionals, as the first is to the third, so is the figure described upon the first to the similar and similarly described figure upon the second (VI. 20, *Cor.* 2). Therefore B C is to C F, as the rectilineal figure A B C is to the rectilineal figure K G H. But B C is to C F, as the parallelogram B E is to the parallelogram E F (VI. 1). Therefore the rectilineal figure A B C is to the rectilineal figure K G H, as the parallelogram B E is to the parallelogram E F (V. 11). But the rectilineal figure A B C is equal to the parallelogram B E (*Const.*). Therefore the rectilineal figure K G H is equal to the parallelogram E F (V. 14). But the parallelogram E F is equal to the rectilineal figure D (*Const.*). Therefore also the rectilineal figure K G H is equal to the rectilineal figure D; and it is similar to the rectilineal figure A B C. Therefore the rectilineal figure K G H has been described similar to the rectilineal figure A B C, and equal to the rectilineal figure D. Q. E. F.

The general proposition intended here may be ennunciated in the following terms: To construct a rectilineal figure of given species and magnitude. The construction amounts to this: On the side of the figure *given in species* construct an equal rectangle, and on one side of this rectangle construct another rectangle equal to the *figure given in magnitude*. The sides of these figures being placed in the same straight line, and a mean proportional between them being found, will give the side of the figure required. By this proposition, while a superficies may be preserved as to quantity, its form may be changed.

PROP. XXVI. THEOREM.

If two similar parallelograms have a common angle, and be similarly situated; they are about the same diagonal.

Let the parallelograms B D and E G be similar and similarly situated, and have the angle D A B common. They are about the same diagonal.



For, if not, let, if possible, the parallelogram B D have its diagonal A L C in a different straight line from A F, the diagonal of the parallelogram E G. Let G F meet A L C in L. Through L draw L K parallel to A D or B C.

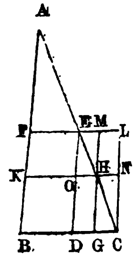
Because the parallelograms B D and K G are about the same diagonal, they are similar to one another (VI. 24). Therefore D A is to A B, as G A is to A K (VI. *Def.* 1). Because B D and E G are similar parallelograms (*Hyp.*), D A is to A B, as G A is to A E. Therefore, G A is to A E, as G A is to A K (V. 11). Because G A has the same ratio to each of the straight lines A E and A K. Therefore, A K is equal to A E (V. 9); the less to the greater, which is impossible. Therefore the parallelogram B D cannot have its diagonal in A L C a different straight line from A F. Wherefore B D and E G are about the same diameter A F C. Therefore, if two similar, &c. Q. E. D.

This proposition is a sort of converse of Prop. XXIV. of this Book, and Prop. XXV. seems to be awkwardly placed between them.

PROP. XXVII. THEOREM.

Of all the parallelograms inscribed in a triangle so as to have one of the angles at the base common to them all, the greatest is that described on half the base.

Let A B C be a triangle, having its base B C bisected in D. Let B E and B H be parallelograms inscribed in it, so as to have the angle A B C of the triangle common to both. The parallelogram B E described on B D half the base, is greater than B H.



Complete the parallelogram B L, and produce G H to M, and K H to N.

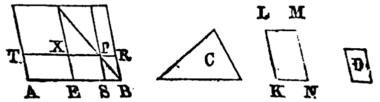
Because B D is equal to D C (*Hyp.*), the parallelogram B E is equal to the parallelogram D L, and the parallelogram BO to the parallelogram D N (I. 36). Because D L is a parallelogram and E C its diagonal, the complement D H is equal to the complement H L (I. 43). Adding equals to equals, the parallelogram B H is equal to the gnomon D N M (I. *Ax.* 2). But the parallelogram D L is greater than the gnomon D N M (I. *Ax.* 9). Therefore the parallelogram B E is greater than the gnomon D N M. But the parallelogram B H has been proved equal to the gnomon D N M. Therefore the parallelogram B E is greater than the parallelogram B H. In the same manner, it may be shown that the parallelogram B E described on half the base B C, is greater than any other parallelogram described on a segment greater or less than half the base. Therefore, of all the parallelograms, &c. Q. E. D.

Corollary.—If on the two segments of a given straight line B C, two parallelograms of the same altitude be described, viz. B H and G N, and one of them G N be similar to a given parallelogram O M or D L, the other parallelogram will be the greatest possible when the segments of the given straight line are equal.

PROP. XXVIII. PROBLEM.

To divide a given straight line into two parts such that of the two parallelograms having the same altitude described upon them, one may be similar to a given parallelogram, and the other equal to a given rectilineal figure not greater than a parallelogram, similar to the given parallelogram, described on half of the given straight line (VI. 27, *Cor.*).

Let A B be the given straight line, D the given parallelogram, and C the given rectilineal figure. It is required to divide A B into two parts, such that of the two parallelograms having the same altitude described upon them, one may be similar to D, and the other equal to C, which must not be greater than a parallelogram similar to D, described on half of A B (VI. 27, *Cor.*).



Bisect A B at E (I. 10), and upon E B describe the parallelogram E F similar and similarly situated to D (*VI. 18*). Complete the *parallelogram* A F. Then the *parallelogram* A G which is equal to the parallelogram E F (I. 36), is either equal to the rectilineal figure C, or greater than it (*Hyp.*). If the parallelogram A G, is equal to the rectilineal figure C, what was required is done; for the straight line A B is divided into two parts at E, such that of the two parallelograms A G and E F of the same altitude, described upon them one E F is similar to the given parallelogram D, and the other A G is equal to the given rectilineal figure C, which is not greater than the parallelogram E F similar to the given parallelogram D, and described on E B, half the given straight line.

But if the parallelogram A G is greater than the rectilineal figure C, the parallelogram E F is also greater than it. Describe the parallelogram K M equal to the excess of E F above C, and similar and similarly situated to the parallelogram D (VI. 25). Because E F and K M are each similar to D (*Const.*). Therefore E F is similar to K M (VI. 21), and greater than it (*Const.*). Wherefore the sides E G and G F of the parallelogram E F, are each greater than their homologous sides K L and L M of the parallelogram K M. From G E, cut off G X equal to L K (I. 3); and from G F, cut off G O equal to L M. Complete the parallelogram O X (I. 31), and O X is equal and similar to K M. But K M is similar to E F (*Const.*). Therefore O X is similar to E F, and it is similarly situated. Therefore O X and E F are about the same diagonal G B (VI. 26). Produce X P to R, and O P to S. The straight line A B is divided at S, as required.

Because E F is equal to C and K M together (*Const.*). But X O has been proved equal to K M. Therefore the remainder the gnomon E R O is equal to the remainder C (I. *Ax.* 3). But it may be shown, as in the last proposition, that the parallelogram T S is equal to the gnomon E R O. Therefore T S is equal to C. Also S R is similar to D (VI. 24). Therefore the straight line A B is divided into two parts at S, such that of the two parallelograms T S and S R having the same altitude described upon them, one S R is similar to the given parallelogram D, and the other T S is equal to C, a given rectilineal figure not greater than a parallelogram E F similar to the given parallelogram D, described on E B half of the given straight line A B. Q. E. F.

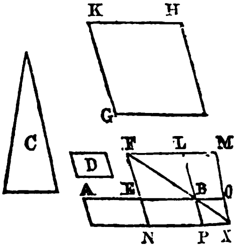
PROP. XXIX. PROBLEM.

To produce a given straight line so that of the two parallelograms of the same altitude described on the whole line thus produced and on the part produced, the one on the part produced may be similar to a given parallelogram, and the other on the whole produced equal to a given rectilineal figure.

Let A B be the given straight line, D the given parallelogram, and C the given rectilineal figure. It is required to produce A B, so that of the two parallelograms of the same altitude described on the whole of A B produced and on the part produced, the one on the part produced may be similar to D, and the other on the whole produced may be equal to C.

Bisect A B at E (I. 10), and upon E B describe the parallelogram E L similar and similarly situated to D (VI. 18). Make the parallelogram G H equal to E L and C together, and similar and similarly situated to D (VI. 25). Because G H is similar to E L (VI. 21), *and greater than it* (*Const.*). Therefore the sides G K and K H of the *parallelogram* G H, are greater than the homologous sides E F and

FL of the parallelogram EL. Produce FL and FE, and make FM equal to KH, and FN to KG (I. 3). Complete the parallelogram MN (I. 31). MN is equal and similar to GH (*Const.*). But GH is similar to EL. Therefore MN is similar to EL. Wherefore EL and MN are about the same diagonal FX. Produce LB to P, and AB to O. The straight line AB is produced to O as required (IV. 26).



Because GH is equal to EL and C together, and GH is equal to MN (*Const.*). Therefore MN is equal to EL and C (I. *Ax.* 1). From these equals, take away the common part EL. Therefore the remainder, the gnomon NOL, is equal to C (I. *Ax.* 3). Because AE is equal to EB, the parallelogram AN is equal to the parallelogram NB (I. 36); that is, to BM (I. 43). To these equals add NO. Therefore the parallelogram AX is equal to the gnomon NOL (I. *Ax.* 2). But the gnomon NOL has been proved equal to C. Therefore also AX is equal to C, and PO is similar to D (VI. 24). Wherefore the given straight line AB has been produced, so that of the two parallelograms AX and PO, of the same altitude described on the whole AO thus produced, and on the part BO produced, the one PO on the part produced is similar to the given parallelogram D, and the other AX on the whole produced is equal to the given rectilineal figure C. Q. E. F.

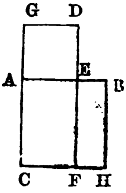
The three preceding propositions are essentially the same as Euclid's three propositions usually given, but the enunciations are very different indeed. The phraseology of Euclid's enunciations is very difficult for a learner to understand, on account of its obscure technicality; and it was therefore deemed adviseable to present these propositions in an improved and intelligible form. This will account for the striking apparent difference, in this respect, between the old editions of Euclid and the present.

PROP. XXX. PROBLEM.

To cut a given straight line in extreme and mean ratio.

Let AB be the given straight line. It is required to cut it in extreme and mean ratio.

Upon AB describe the square BC (I. 46), and produce AC to G, so that AD described on the part produced may be a square similar to BC, and CD a rectangle equal to BC (VI. 29).



Because BC is equal to CD, and the part CE common to both. From each take CE, and the remainder BF is equal to the remainder AD (I. *Ax.* 3). Because BF and AD are equiangular (*Const.*) and equal. Therefore their sides about the equal angles at E are reciprocally proportional (VI. 14). Wherefore FE is to ED, as AE *is to EB.* But FE is equal to AC (I. 34), that is, to *AB (Def. 30);* and ED is equal to AE. Therefore *BA is to AE,* as AE is to EB. But AB is greater than AE (Const.). *Therefore AE is greater* than EB (V. 14). Wherefore the straight ine A B is cut in extreme and mean ratio at the point E (VI. *Def.* 3). . E. F.

OTHERWISE.—Let A B be the given straight line. It is required to ut it in extreme and mean ratio.

Divide A B at the point C, so that the rectangle ontained by A B and B C, may be equal to the quare of A C (II. 11).

Because the rectangle A B.B C is equal to the square of A C. Thereore B A is to A C, as A C is to C B (VI. 17). Wherefore A B is cut in xtreme and mean ratio at C (VI. *Def.* 3). Q. E. F.

Corollary 1.—If a straight line be divided in extreme and mean ratio, and a part be cut off from the greater segment equal to the less, the greater segment is also divided in extreme and mean ratio; and by continuing this process, the division into extreme and mean ratio may be carried on to infinity, the parts continually growing smaller.

Corollary 2.—If a straight line be divided in extreme and mean ratio, and if it be produced until the part produced be equal to the greater segment, the whole line thus produced is also divided in extreme and mean ratio; and by continuing this process, the division into extreme and mean ratio may also be carried on to infinity, the parts continually growing larger.

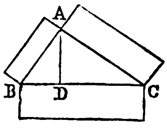
Corollary 3.—If the hypotenuse of a right-angled triangle be divided in extreme and mean ratio, by the perpendicular drawn to it from the right angle; the less side of the triangle is equal to the greater segment of the hypotenuse; and *conversely.*

PROP. XXXI. THEOREM.

n right-angled triangles, the rectilineal figure described upon the side opposite to the right angle is equal to the similar and similarly described rectilineal figures upon the sides containing the right angle.

Let A B C be a right-angled triangle, having the right angle B A C. he rectilineal figure described upon B C is equal to the similar and imilarly described figures upon B A and A C.

Draw the perpendicular A D (I. 12).



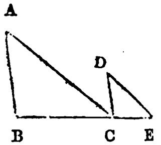
Because in the right-angled triangle A B C, A D is drawn from the ight angle at A perpendicular to the base B C, the triangles A B D and D C are similar to the whole triangle A B C, and to ne another (VI. 8). Because the triangle A B C is imilar to A D B. Therefore C B is to B A, as B A is o B D (VI. 4), Because these three straight lines re proportionals, the first is to the third, as the gure described upon the first is to the similar and imilarly described figure upon the second (VI. 20, or. 2). Therefore C B is to B D, as the figure described upon C B is o the similar and similarly described figure upon B A. Wherefore, nversely, D B is to B C, as the figure described upon B A is to that escribed upon B C (V. B). For the same reason, D C is to C B, as he figure described upon C A is to that described upon C B. Therefore D and D C together are to B C, as the figures described upon B A nd A C together are to that described upon B C (V. 24). But B D and C together are equal to B C (I. *Ax.* 9). Therefore the figure described n *B C is equal to* the similar and similarly described figures on B A and *C (V. A).* Wherefore, in right-angled triangles, &c. Q. E. D.

The celebrated Pythagorean theorem, the 47th proposition of the first Book, is only a particular case of this proposition, namely, when the similar and similarly described figures are squares. Here, then, is an additional and independent demonstration or confirmation of that proposition.

PROP. XXXII. THEOREM.

If two triangles, which have two sides of the one proportional to two sides of the other, be joined at one angle, so as to have their homologous sides parallel to one another and proceeding from their vertices in the same direction; the remaining sides shall be in the same straight line.

Let A B C and D C E be two triangles which have the two sides B A and A C proportional to the two sides C D and D E; that is, B A to A C, as C D to D E. Also let A B be parallel to D C, and A C to D E. The remaining sides B C and C E shall be in the same straight line.



Because A B is parallel to D C, and the straight line A C meets them the alternate angles B A C and A C D are equal (I. 29). For the same reason, the angle C D E is equal to the angle A C D. Therefore B A C is equal to C D E (I. *Ax.* 1). Because the triangles A B C and D C E have the angle at A equal to the angle at D, and the sides about these angles proportionals, that is, B A to A C, as C D to D E (*Hyp.*) Therefore the triangle A B C is equiangular to D C E (VI. 6); and the angle A B C is equal to the angle D C E. But the angle B A C was proved to be equal to A C D. Therefore the whole angle A C E is equal to the two angles A B C and B A C (I. *Ax.* 2). To each of these equals add the common angle A C B. Therefore the two angles A C E and A C B are equal to the three angles A B C, B A C and A C B (I. *Ax.* 2). But the three angles A B C, B A C and A C B are equal to two right angles (I. 32). Therefore the two angles A C E and A C B are equal to two right angles (I. *Ax.* 1). Because at the point C, in the straight line A C, the two straight lines B C and C E on the opposite sides of it, make the adjacent angles A C E and A C B equal to two right angles. Therefore B C and C E are in the same straight line (I. 14). Wherefore, if two triangles, which have two sides of the one proportional, &c. Q. E. D.

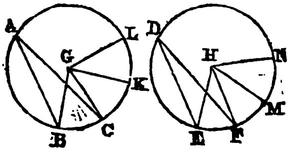
Dr. Simson has given the following demonstration of this proposition in his notes: Let A E F, and F H C (see fig. to the 24th proposition of this Book), be the triangles placed as required. Draw C K parallel to F H (I. 31), and let it meet E F produced in K.

Because A E and K C are each parallel to F H (*Const.* and *Hyp.*), they are parallel to each other (I. 30). Therefore the alternate angles A E F and F K C are equal, and A E is to E F as F H is to H C, that is, as C K to K F. Wherefore the triangles A E F, and C K F are equiangular (VI. 6), and the angle A F E is equal to the angle C F K. But E F K is a straight line (*Const.*) Therefore A F and F C are in the same straight line (I. 14). Q. E. D.

PROP. XXXIII. THEOREM.

In equal circles, the angles, either at the centres or the circumferences, have the same ratio to one another, as the arcs on which they stand, so also have the sectors.

Let A B C and D E F be equal circles; and let B G C and E H F be angles at their centres, and B A C and E D F angles at their circumferences. The arc B C is to the arc E F, as the angle B G C is to the angle E H F; and as the angle B A C is to the angle E D F. Also, the arc B C is to the arc E F, as the sector B G C is to the sector E H F.

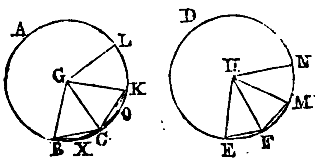


Take any number of arcs C K and K L, each equal to B C; and any number F M and M N, each equal to E F. Join G K, G L, H M and H N.

Because the arcs B C, C K and K L are all equal, the angles B G C, C G K and K G L are also all equal (III. 27). Therefore what multiple soever the arc B L is of the arc B C, the same multiple is the angle B G L of the angle B G C. For the same reason, whatever multiple the arc E N is of the arc E F, the same multiple is the angle E H N of the angle E H F. If the arc B L be equal to the arc E N, the angle B G L is equal to the angle E H N (III. 27); if the arc B L be greater than the arc E N, the angle B G L is greater than the angle E H N; and if less, less. But there are four magnitudes, the two arcs B C and E F, and the two angles B G C and E H F; and of the arc B C, and the angle B G C, have been taken any equimultiples whatever, viz., the arc B L, and the angle B G L; and of the arc E F, and of the angle E H F, any equimultiples whatever, viz., the arc E N, and the angle E H N. And it has been proved, that if the arc B L be greater than the arc E N; the angle B G L is greater than the angle E H N. if equal, equal; and if less, less. Therefore the arc B C is to the arc E F, as the angle B G C is to the angle E H F (V. *Def.* 5). But the angle B G C is to the angle E H F, as the angle B A C is to the angle E D F (V. 15); each being double of each (III. 20). Therefore the arc B C is to the arc E F, as the angle B G C is to the angle E H F; and as the angle B A C is to the angle E D F.

Next, the arc B C is to the arc E F, as the sector B G C is to the sector E H F.

Join B C and C K; and in the arcs B C and C K take any points X and O. Join B X, X C, C O and O K.



Because in the triangles G B C and G C K, the two sides B G and G C are equal to the two sides C G and G K, each to each, and they contain equal angles (*Const.*). Therefore the base B C is equal to the base C K (I. 4), and the triangle G B C to the triangle G C K. Because the arc B C is equal to the arc C K, the remaining part of the whole circumference of the circle A B C, is equal to the remaining part of the whole circumference of the same circle (I. *Ax.* 3). Therefore the angle B X C is equal to the angle C O K (III. 27). Wherefore the segment B X C is similar to the segment C O K (III. *Def.* 11). But they are upon equal straight lines, B C and C K; and similar segments of circles upon equal straight lines *are equal to one another* (III. 24). Therefore the segment B X C is *equal to the segment* C O K. But the triangle B G C was proved to equal to the triangle C G K. Therefore the whole, the sector B G C, equal to the whole, the sector C G K. For the same reason, the sector K G L is equal to each of the sectors B G C and C G K. In the same manner, the sectors E H F, F H M and M H N may be proved equal one another. Therefore, what multiple soever the arc B L is of the arc B C, the same multiple is the sector B G L of the sector B G C. For the same reason, whatever multiple the arc E N is of the arc E F, the same multiple is the sector E H N of the sector E H F. If the arc B L be equal to the arc E N, the sector B G L is equal to the sector E H N; if the arc B L be greater than the arc E N, the sector B G L is greater than the sector E H N; and if less, less. But there are four magnitudes, the two arcs B C and E F, and the two sectors B G C and E H F; and of the arc B C and the sector B G C, the arc B L and the sector B G L are any equimultiples whatever; and of the arc E F and the sector E H F, the arc E N and the sector E H N are any equimultiples whatever. And it has been proved, that if the arc B L be greater than the arc E N, the sector B G L is greater than the sector E H N; if equal, equal; and if less, less. Therefore, the arc B C is to the arc E F, as the sector B G C is to the sector E H F (V. *Def.* 5). Wherefore, in equal circles, &c. Q. E. D.

Corollary 1.—**In the same circle, the angles, either at the centres or circumferences, are to one another, as the arcs on which they stand; so also are the sectors.**

Corollary 2.—**In the same circle, any angle is to a right angle, as the arc on which it stands is to a quadrant; and any angle is to four right angles, as the arc on which it stands is to the whole circumference.**

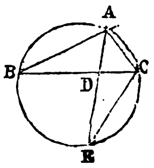
The following propositions, marked B, C, and D, were added to this Book by Dr. Simson, because they are frequently made use of by geometers.

PROP. B. THEOREM

If any angle of a triangle be bisected by a straight line which likewise cuts the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square of the straight line which bisects the angle.

Let A B C be a triangle, and let the angle B A C be bisected by the straight line A D. The rectangle B A.A C is equal to the rectangle B D.D C, together with the square of A D.

Describe the circle A C B about the triangle (IV. 5). Produce A D to meet the circumference in E, and join E C.

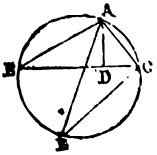


Because the angle B A D is equal to the angle C A E (*Hyp.*); and the angle A B D to the angle A E C (III. 21), for they are in the same segment. Therefore the triangles A B D and A E C are equiangular to one another (I. 32); and B A is to A D, as E A is to A C (VI. 4). But the rectangle B A.A C is equal to the rectangle E A.A D (VI. 16); that is, to the rectangle E D.D A, together with the square of A D (II. 3). Because the rectangle E D.D A is equal to the rectangle B D.D C (III. 35). Therefore the rectangle B A.A C is equal to the rectangle B D.D C, together with the square of A D (I. *Ax.* 1). Wherefore, if an angle, &c. Q. E. D.

PROP. C. THEOREM.

If from any angle of a triangle a straight line be drawn perpendicular to the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

Let A B C be a triangle, and A D the perpendicular drawn from the angle A to the base B C. The rectangle B A.A C is equal to the rectangle contained by A D and the diameter of the circle described about the triangle.



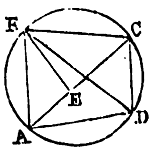
Describe the circle A C B about the triangle (IV. 5); draw the diameter A E, and join E C.

Because the right angle B D A is equal to the angle E C A in a semicircle (III. 31); and the angle A B D to the angle A E C in the same segment (III. 21). Therefore the triangles A B D and A E C are equiangular, and B A is to A D, as E A is to A C (VI. 4). Wherefore the rectangle B A.A C is equal to the rectangle E A.A D (VI. 16). Therefore, if from any angle, &c. Q. E. D.

PROP. D. THEOREM.

The rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle, is equal to both the rectangles contained by its opposite sides.

Let A B C D be any quadrilateral figure inscribed in a circle, and A C and B D its diagonals. The rectangle A C.B D is equal to the two rectangles A B.C D, and A D.B C.



Make the angle A B E equal to the angle D B C (I. 23). Because the angle A B E is equal to the angle D B C. To each of these equals add the common angle E B D. Therefore the angle A B D is equal to the angle E B C. But the angle B D A is equal to the angle B C E, because they are in the same segment (III. 21). Therefore the triangle A B D is equiangular to the triangle B C E; and B C is to C E, as B D is to D A (VI. 4). Wherefore the rectangle B C.A D is equal to the rectangle B D.C E (VI. 16).

Again, because the angle A B E is equal to the angle D B C, and the angle B A E to the angle B D C (III. 21). Therefore the triangle A B E is equiangular to the triangle B C D; and B A is to A E, as B D is to D C. Wherefore the rectangle B A.D C is equal to the rectangle B D.A E. but the rectangle B C.A D has been shown to be equal to the rectangle B D.C E. Therefore the rectangles B C.A D and B A.D C are together equal (I. *Ax.* 2) to the rectangles B D.C E and B D.A E, that is, to the whole rectangle B D.A C. Therefore the whole rectangle A C.B D is equal to the two rectangles A B.D C, and A D.B C (II. 1). Therefore the rectangle, &c. Q. E. D.

Corollary.—The sum of the chords drawn from the extremities of any arc of a circle to any point in the remaining part of the circumference, is to the chord drawn from the middle of the arc to the same point, as the chord of the whole arc is to *the chord of half* the arc.

This Proposition D is a Lemma of Cl. Ptolemæus, in page 9 of his Μεγάλη Σύνταξις, or "*Great Construction.*"

BOOK XI.

DEFINITIONS.

I.

A SOLID is that which hath length, breadth, and thickness.

A solid is extension in any three directions, uniform or variable; and strictly speaking, signifies a definite portion of space.

II.

That which bounds a solid is a superficies or surface.

This definition simply signifies that the boundaries of solids are surfaces.

III.

A straight line is perpendicular, or at right angles, to a plane, when it makes right angles with every straight line meeting it in that plane.

IV.

A plane is perpendicular to a plane, when the straight lines drawn in one of the planes perpendicular to the common section of the two planes, are perpendicular to the other plane.

The *common section* of two planes is the line in which the one cuts the other, when they intersect or cross each other.

V.

The inclination of a straight line to a plane, is the acute angle contained by that straight line, and another drawn from the point in which it meets the plane, to the point in which a perpendicular to the plane drawn from any point of the first line above the plane, meets the same plane.

The meaning of this definition will be more easily understood, by conceiving a plane to pass through the straight line, cutting the plane at right angles. The angle between the straight line and the common section of these planes is the *inclination* of the straight line to the plane.

VI.

The inclination of one plane to another is the acute angle contained by two straight lines drawn from any point in their common section at right angles to it, one upon each plane.

The meaning of this definition will be best understood by conceiving a plane to cut both planes at right angles to their common section. The angle between the common sections of this third plane with the other two is their *inclination*.

VII.

Two planes are said to have the same inclination to one another which two other planes have, when their angles of inclination are equal.

VIII.

Parallel planes are such as do not meet one another though produced ever so far in all directions.

The meaning of this definition is that that the space between the planes is always of the same width.

IX.

A solid angle is that which is made by the meeting of more than two plane angles in one point, but which are not in the same plane.

The term *solid*, here applied to an angle, merely indicates that the angle described belongs to a solid figure, or one that has length, breadth, and thickness. A solid angle does not enclose space. The *vertex* of a solid angle is the point where all its plane angles meet.

X.

Equal and similar solid figures are such as are contained by similar planes equal in number, magnitude, and inclination to one another.

Dr. Simson has in his edition omitted this definition, on the ground that it is a theorem and not a definition.

XI.

Similar solid figures are such as have all their solid angles equal, each to each, and are contained by the same number of planes similarly situated.

The planes containing the solid angles of any solid figure are similar and similarly situated to the planes containing the corresponding solid angle in another solid figure, only when the vertices of these solid angles being made to coincide, and a plane of the one applied to the corresponding plane of the other, the remaining planes of the one coincide with the remaining planes of the other, each to each.

XII.

A pyramid is a solid figure contained by planes that are constituted between one plane and a point above it in which they meet.

A pyramid may be defined as the solid figure formed by a solid angle and a plane intersecting all its plane angles at any distance from its vertex. This plane is called *the base* of the pyramid.

XIII.

A prism is a solid figure contained by plane figures, of which two that are opposite, are equal, similar and parallel to one another; and the others are parallelograms.

The opposite ends or faces of a prism are generally called its *bases*; although the term base is sometimes applied to any parallelogram on which it is supposed to to stand. The parallelograms are generally called the sides of the prism.

Pyramids and prisms are called *triangular*, *quadrangular*, *pentagonal*, *polygonal*, &c., according as their bases are *triangles*, *quadrangles*, *pentagons*, *polygons*, &c. A prism is called *right*, when its sides are *rectangles*; *oblique*, when otherwise.

XIV.

A sphere is a solid figure described by the revolution of a semicircle about its diameter, which remains unmoved.

A sphere may be defined as a solid figure bounded by one surface of such a kind that all straight lines, drawn from a *certain point* within the solid to its superficies, are equal to one another.

XV.

The axis of a sphere is the fixed straight line about which the semicircle revolves.

Any diameter of a sphere may be made, or supposed to be, an axis of revolution.

XVI.

The centre of a sphere is the same with that of the generating semi circle.

> **The *certain point* within the sphere, from which all the equal straight lines are drawn to the superficies, is called the centre.**

XVII.

The diameter of a sphere is any straight line which passes through the centre, and is terminated both ways by the superficies of the sphere.

> **The straight line drawn from the centre to the superficies of a sphere is called its radius**

XVIII.

A cone is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle, that side remaining fixed.

If the fixed side be equal to the other side containing the right angle, the cone is called a right-angled cone; if it be less than the other side, an obtuse-angled cone; and if greater, an acute-angled cone.

> **A cone may be defined as a solid figure bounded by a circle and a superficies terminating in a point, of such a kind, that all straight lines drawn in it from that point to the circumference of the circle are equal to one another. This point is called the *vertex* of the cone.**

XIX.

The axis of a cone is the fixed straight line about which the triangle revolves.

> **The *axis* of a cone is the straight line drawn from its vertex to the centre of its base.**

XX.

The base of a cone is the circle described by the revolving leg of the right angle.

> **The base of a cone is the circle which forms one of its boundaries.**

XXI.

A cylinder is a solid figure described by the revolution of a rectangle about one of its sides which remains fixed.

> **A cylinder may be defined as a solid figure bounded by two opposite equal and parallel circles, and a superficies of such a kind that all straight lines drawn in it between their circumferences parallel to the straight lines joining their centres, are equal to one another.**

XXII.

The axis of a cylinder is the fixed straight line about which the rectangle revolves.

> **The axis of a cylinder is the straight line which joins the centres of its bases.**

XXIII.

The bases of a cylinder are the circles described by the two revolving opposite sides of the rectangle.

> **The ends or *bases* of a cylinder are the two opposite, equal and parallel circles which form two of its boundaries.**

XXIV.

Similar cones and cylinders are those which have their axes and the *diameters of their* bases proportionals.

A.

lelopiped is a solid figure contained by six quadrilateral figures, ;very opposite two are parallel.

llelopiped is a prism of which the bases are parallelograms. If the bases 'a parallelopiped be rectangles, it is called *right*, if otherwise *oblique*.

B.

ıedron is any solid figure bounded by plane figures. If these res are all equal and similar, the polyhedron is called *regular*.

XXV.

hedron, or cube is a solid figure contained by six equal squares.

is a right parallelopiped of which the sides are squares.

XXVI.

ıedron is a solid figure contained by four equal and equilateral

XXVII.

hedron is a solid figure contained by eight equal and equilateral

XXVIII.

ahedron is a solid figure contained by twelve equal pentagons equilateral and equiangular.

XXIX.

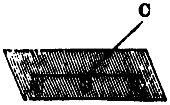
ahedron is a solid figure contained by twenty equal and equiıngles.

e preceding definitions relate only to the *five regular polyhedrons*, or *five* ır *bodies*, as they are called; because no greater number than these five can . The *irregular* polyhedrons are innumerable.

PROP. I. THEOREM.

't of a straight line cannot be in a plane and another part out of it.

possible, let A B, part of the straight line A B C, be in a plane, ırt B C be out of it.



› the straight line A B is in the plane, it can ›d in the plane (I. *Post.* 1). Let A B be to D; and let any plane be made to pass ıe straight line A D, and turn about A D ss through the point C.

ı the points B and C are in this plane, the ne (I *Def.* 7) B C is in it. Therefore the two straight lines . A B D in the same plane, have a common segment A B (I. which is impossible. Therefore one part of a straight line, &c.

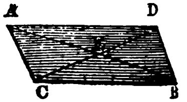
PROP. II. THEOREM.

ght lines which cut one another are in one plane; and three 'ht lines which meet one another in three points, are in one plane.

straight lines A B and C D cut one another in E; A B and

CD are in one plane: and let three straight lines A B, B C, and CD meet one another in three points B, C and E; they are also in one plane.

Let any plane pass through the straight line A B, and let the plane be turned about A B until it pass through the point C. Because the points E and C are in this plane, the straight line CD is in it (I. *Def.* 7, and XI. 1). But the straight line A B is in the same plane (*Const.*). Therefore the straight lines A B and C D are in one plane. Again, because the points B and C are in the same plane with the point E in A B, the straight line B C is in this plane (I. *Def.* 7). But it has been proved that the straight lines A B and C D are in it. Therefore the three straight lines A B, B C, and C D are in one plane. Wherefore two straight lines which cut one another, &c. Q. E. D.

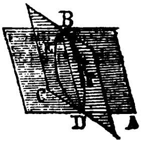


PROP. III. THEOREM.

If two planes cut one another, their common section is a straight line.

Let two planes A B and B C cut one another, and let the line B D be their common section. B D is a straight line.

If B D be not a straight line, from the point D to B (I. *Post.* 1), draw, in the plane A B, the straight line D E B; and in the plane B C, the straight line D F B. Because the two straight lines D E B and D F B have the same extremities B and D, and do not coincide, they include a space between them; which is (I. *Ax.* 10) impossible. Therefore B D, the common section of the planes A B and B C, must be in a straight line. Wherefore, if two planes, &c. Q. E. D.



In definitions fourth and sixth of this Book, Euclid has tacitly assumed the truth of this proposition, and it seems almost impossible to do otherwise.

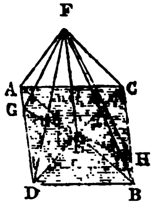
PROP. IV. THEOREM.

If a straight line stand at right angles to each of two straight lines at the point of their intersection, it is at right angles to the plane in which they are.

Let the straight line E F stand at right angles to each of the straight lines A B and C D, at E the point of their intersection. E F is at right angles to the plane of A B and C D, that is, the plane in which they are.

Make E A, E B, E C, and E D all equal to one another (I. 3). Through E draw, in the plane of A B and C D, any straight line G H. Join A D and C B, and let G H meet them in G and H. From any point F, in E F, draw F A, F G, F D, F C, F H, and F B.

Because the two sides A E and E D are equal to the two sides B E and E C, each to each, and they contain equal angles (I. 15) A E D and B E C, the base A D is equal (I. 4) to the base B C, and the angle D A E to the angle E B C. But the angle A E G is equal (I. 15) to the angle B E H. Therefore the two triangles A E G and B E H have two angles of the one equal to two angles of the other, each to each; and the sides A E and E B, adjacent to the equal angles, equal to one another. Wherefore their other sides are equal (I. 26). Therefore G E is equal to E H, and A G to B H. Because A E is equal to E B, F E common, and the angle A E F equal to the angle B E F (*Hyp.*). Therefore the base A F is equal (I. 4) to the base F B. For the same reason, C F is equal to F D. Because A D is equal to B C, and A F to F B, the two sides F A and A D are equal to the two F B and B C, each to each. But the base D F has been proved equal to the base F C. Therefore the angle F A D is equal (I. 8) to the angle F B C. But it has been proved that G A is equal to B H, and also A F to F B. Therefore F A and A G are equal to F B and B H, each to each, and the angle F A G has been proved equal to the angle F B H. Wherefore the base G F is equal (I. 4) to the base F H. But it has been proved that G E is equal to E H, and E F is common. Therefore G E and E F are equal to H E and E F, each to each; and the base G F is equal to the base F H. Wherefore the angle G E F is equal (I. 8) to the angle H E F; and each of these angles is a right (I. *Def.* 10) angle. Therefore the straight line F E makes right angles with G H, that is, with any straight line drawn through E in the plane of A B and C D. In like manner, it may be proved, that F E makes right angles with every other straight line meeting it in that plane. But a straight line is at right angles to a plane when it makes right angles with every straight line meeting it in that plane (XI. *Def.* 3). Therefore E F is at right angles to the plane of A B and C D. Wherefore, if a straight line stand at right angles, &c. Q. E. D.



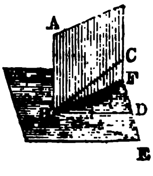
PROP. V. THEOREM.

If three straight lines meet all in one point, and a straight line stands at right angles to each of them at that point; these three straight lines are in the same plane.

Let the straight line A B stand at right angles to each of the straight lines B C, B D, and B E at the point B where they meet. The straight lines B C, B D, and B E are in the same plane.

If they be not in the same plane, let, if possible, B D and B E be in one plane, and B C out of it. Let a plane pass through A B and B C, and let the straight line B F (XI. 3) be the common section of this plane, and the plane of B D and B E.

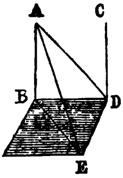
Because the three straight lines AB, B C, and B F are all in one plane (*Hyp.*), viz., that which passes through A B and B C: and A B stands at right angles to each of the straight lines B D and B E. Therefore A B is at right angles (XI. 4) to the plane in which they are. Wherefore A B makes right angles (XI. *Def.* 3) with every straight line meeting it in that plane. But B F, which is in that plane, meets it. Therefore the angle A B F is a right angle. But the angle A B C (*Hyp.*) is also a right angle. Therefore the angle A B F is equal to the angle A B C, and they are both in the same plane, which is (I. *Ax.* 9) impossible. Therefore the straight line B C is not out of the plane of B D and B E. Wherefore the three straight lines B C, B D, *and B E are in the* same plane. Therefore, if three straight lines meet *all in one point, &c.* Q. E. D.



PROP. VI. THEOREM.

If two straight lines be at right angles to the same plane, they are parallel.

Let the straight lines A B and C D be at right angles to the same plane. A B is parallel to C D.



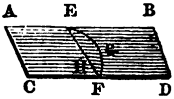
Let the straight lines meet the plane in the points B and D. Join B D, and draw (I. 11) D E at right angles to it in the same plane. Make (I. 3) D E equal to A B, join B E, A E and A D.

Because A B is perpendicular to the plane, it makes right (XI. *Def.* 3) angles with every straight line meeting it in that plane. But B D and B E in that plane each meet A B. Therefore each of the angles A B D and A B E is a right angle. For the same reason, each of the angles C D B and C D E is a right angle. Because A B is equal to D E, and B D common, the two sides A B and B D are equal to the two E D and D B, each to each; and they contain right angles. Therefore, the base A D is equal (I. 4) to the base B E. Again, because A B is equal to D E, and B E to A D; A B and B E are equal to E D and D A, each to each; and, in the triangles A B E and E D A, the base A E is common. Therefore the angle A B E is equal (I. 8) to the angle E D A. But A B E is a right angle. Therefore E D A is also a right angle, and E D perpendicular to D A. But it is also perpendicular to each of the two B D and D C. Therefore E D is at right angles to each of the three straight lines B D, D A, and D C at the point D where they meet. Wherefore (XI. 5) these three straight lines are all in the same plane. But A B is in the plane of B D and D A (XI. 2), because any three straight lines which meet in three points are in one plane. Therefore A B, B D, and D C are in one plane: and each of the angles A B D and B D C is a right angle. Therefore A B is parallel (I. 28) to C D. Wherefore, if two straight lines be at right angles, &c. Q. E. D.

PROP. VII. THEOREM.

If two straight lines be parallel, the straight line drawn from any point in the one, to any point in the other, is in the same plane with the parallels.

Let AB and CD be parallel straight lines, and E any point in the one, and F any point in the other. The straight line which joins E and F is in the same plane with the parallels.



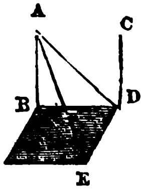
If this straight line be not in the same plane with them, let it, if possible, be out of the plane, as E G F. In the plane A D of the parallels, draw the straight line E H F from E to F.

Because the two straight lines E H F and E G F have the same extremities and do not coincide, they include a space between them, which is (I. *Ax.* 10) impossible. Therefore the straight line joining the points E and F is not out of the plane of the parallels *AB and CD.* Wherefore it is in that plane. Therefore, if two straight *lines be parallel, &c.* Q. E. D.

PROP. VIII. THEOREM.

If two straight lines be parallel, and one of them be at right angles to a plane; the other is also at right angles to the same plane.

Let AB and CD be two parallel straight lines, and let one of them AB be at right angles to a plane: the other CD is also at right angles to the plane.



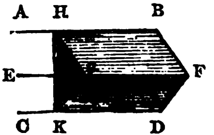
Let AB and CD meet the plane in the points B and D, and join BD. The straight lines (XI. 7) AB, CD, and BD are in one plane. In the plane BDE draw (I. 11) DE at right angles to BD, make (I. 3) DE equal to AB; and join BE, AE, and AD.

Because AB is perpendicular to the plane, it is perpendicular to BE and BD (XI. *Def.* 3). Therefore each of the angles ABD and ABE is a right angle. Because the straight line BD meets the parallel straight lines AB and CD, the angles ABD and CDB are together equal (I. 29) to two right angles. But ABD is a right angle. Therefore also CDB is a right angle, and CD is perpendicular to BD. But it may be proved as in the sixth proposition, that ED is perpendicular to DA; and it is also perpendicular to (*Const.*) BD. Therefore ED is perpendicular (XI. 4) to the plane of BD and DA; and (XI. *Def.* 3) makes right angles with every straight line meeting it in that plane. But DC is in the plane of BD and DA, because all three are in the plane of the parallels AB and CD. Therefore ED is at right angles to DC, and CD to DE. But CD is also at right angles to DB. Therefore CD is at right angles to the two straight lines DE and DB at the point of their intersection D. Wherefore it is at right angles (XI. 4) to the plane of DE and DB, that is the same plane to which AB is at right angles. Therefore, if two straight lines, &c. Q. E. D.

PROP. IX. THEOREM.

Two straight lines which are each of them parallel to the same straight line, but not in the same plane with it, are parallel to one another.

Let AB and CD be each of them parallel to EF, but not in the same plane with it. AB is parallel to CD.



In EF, take any point G, and from it (I. 11) draw, in the plane of EF and AB, the straight line GH at right angles to EF; and in the plane of EF and CD, GK at right angles to EF.

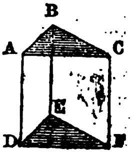
Because EF is perpendicular both to GH and GK, it is perpendicular (XI. 4) to the plane HGK passing through them. But EF is parallel to AB (*Hyp.*). Therefore AB is at right angles (XI. 8) to the plane HGK. For the same reason, CD is at right angles to the plane HGK. Therefore AB and CD are each of them at right angles to the plane HGK. But if two straight lines are at right angles to the same plane, they are parallel (*XI.* 6). Therefore AB is parallel to CD. Wherefore, two *straight lines, &c.* Q. E. D.

PROP. X. THEOREM.

If two straight lines meeting one another be parallel to two others which meet one another, but are not in the same plane with the first two; the first two and the other two contain equal angles.

Let the two straight lines A B and B C, meeting one another, be parallel to the two straight lines D E and E F, which meet one another, and are not in the same plane with A B and B C. The angle A B C is equal to the angle D E F.

Make B A, B C, E D, and E F all equal to one another. Join A D, C F, B E, A C, and D F.



Because B A is equal and parallel to E D, therefore A D is (I. 33) equal and parallel to B E. For the same reason, C F is equal and parallel to B E. Therefore A D and C F are each equal and parallel to B E. But straight lines that are parallel to the same straight line, but not in the same plane with it, are parallel (XI. 9) to one another. Therefore A D is parallel to C F. Because A D is equal (I. *Ax.* 1) and parallel to C F, and A C and D F join them towards the same parts. Therefore (I. 33) A C is equal and parallel to D F. Because A B and B C are equal to D E and E F, each to each, and the base A C to the base D F. Therefore the angle A B C is equal (I. 8) to the angle D E F. Wherefore, if two straight lines, &c. Q. E. D.

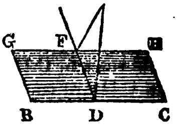
In this proposition, the straight lines which form the corresponding legs of the two different angles are understood to be drawn in the same direction; otherwise, the angles instead of being equal would be supplementary.

PROP. XI. PROBLEM.

To draw a straight line perpendicular to a plane, from a given point above it.

Let A be the given point above the plane B H. It is required to draw from the point A a straight line perpendicular to the plane B H.

In the plane B H draw any straight line B C, and in the plane passing through B C and A, from the point A draw (I. 12) A D perpendicular to B C. If A D be also perpendicular to the plane B H, what was required is done. But if it be not, from the point D draw (I. 11) in the plane B H, the straight line D E at right angles to B C; and in the plane passing through D E and A, from the point A, draw A F perpendicular to D E. A F is perpendicular to the plane B H. Through F, draw (I. 31) G H parallel to B C.

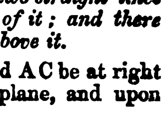


Because B C is at right angles to E D and D A (*Const.*), B C is at right angles (XI. 4) to the plane of E D and D A. But G H is parallel to B C; and if two straight lines be parallel, one of which is at right angles to a plane, the other is also at right (XI. 8) angles to the plane. Therefore G H is at right angles to the plane of E D and D A; and is perpendicular (XI. *Def.* 3) to every straight line meeting it in that plane. But A F, which is in the plane of E D and D A, meets it. Therefore G H is perpendicular to A F; and A F to G H. But A F is perpendicular to D E (*Const.*). Therefore A F is perpendicular to each of the straight lines G H and D E. But if a straight line stand at right angles to each of two straight lines at the point of their intersection, it is also at right angles (XI. 4) to the plane in which they are, that is, the plane B H. Therefore A F is perpendicular to the plane B H. Wherefore, from the given point A, above the plane B H, the straight line A F is drawn perpendicular to it. Q. E. F.

PROP. XII. PROBLEM.

To draw a straight line at right angles to a given plane, from a point given in the plane.

Let A be the given point in the plane. It is required to draw a straight line from the point A at right angles to the plane.



From any point B above the plane, draw (XI. 11) B C perpendicular to it; and from A draw (I. 31) A D parallel to B C. A D is at right angles to the plane.

Because A D and C B are two parallel straight lines, and one of them B C is at right angles to the given plane, the other A D is also (XI. 8) at right angles to it. Therefore a straight line A D has been drawn at right angles to a given plane, from a point A given in it. Q. E. F.

PROP. XIII. THEOREM.

From the same point in a given plane, there cannot be two straight lines drawn perpendicular to the plane, upon the same side of it; and there can be but one perpendicular to a plane, from a point above it.

For, if it be possible, let the two straight lines A B and A C be at right angles to a given plane from the same point A in the plane, and upon the same side of it.

Let a plane pass through B A and A C; and let the straight (I. 3) line D E be their common section. The straight lines A B, A C, and D E are in one plane.

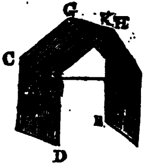
Because C A is at right angles to the given plane it makes right angles (XI. *Def.* 3) with every straight line meeting it in that plane. But D E, which is in that plane, meets C A. Therefore C A E is a right angle. For the same reason, B A E is a right angle. Wherefore the angle C A E is equal (I. *Ax.* 11) to the angle B A E; and they are in one plane, which is impossible (I. *Ax.* 9). Therefore two perpendiculars cannot be drawn from the same point in a plane, on the same side of it. Also, from a point above a plane, there can be but one perpendicular to it. For, if there could be two, they would be parallel (XI. 6) to one another, which is absurd. Therefore, from the same point, &c. Q. E. D.

PROP. XIV. THEOREM.

Planes to which the same straight line is perpendicular, are parallel.

Let the straight line A B be perpendicular to each of the planes C D *and E F. These planes* are parallel.

If not, they shall meet one another when produced. Let them be produced and meet; and let the straight line G H be their common section. In G H, take any point K, and join A K and B K.



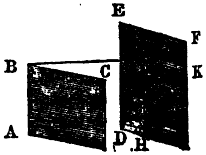
Because A B is perpendicular to the plane E F it is perpendicular (XI. *Def.* 3) to the straight line B K meeting it in that plane. Therefore A B K is a right angle. For the same reason, B A K is a right angle. Therefore the two angles A B K and B A K of the triangle A B K, are equal to two right angles, which is (I. 17) impossible. Therefore the planes C D and E F, though produced, do not meet one another; that is (XI. *Def.* 8), they are parallel. Therefore planes, &c. Q. E. D.

PROP. XV. THEOREM.

Two planes are parallel, if two straight lines which meet each other in the one plane, be parallel to two straight lines which meet each other in the other plane.

Let A B and B C, two straight lines which meet each other, be parallel to two straight lines D E and E F, which meet each other, but are not in the same plane with A B and B C. The plane of A B and B C is parallel to the plane of D E and E F.

From the point B draw B G perpendicular (XI. 11) to the plane of D E and E F, and let it meet that plane in G. Through G draw G H parallel (I. 31) to E D, and G K parallel to E F.



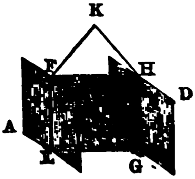
Because B G is perpendicular to the plane of D E and E F, it makes (XI. *Def.* 3) right angles with the straight lines G H and G K meeting it in that plane. Therefore each of the angles B G H and B G K is a right angle. Because B A and G H are each parallel to D E, and are not both in the same plane with it. B A is parallel to G H (XI. 9). Because the angles G B A and B G H are together equal (I. 29) to two right angles; and B G H is a right angle. Therefore also G B A is a right angle, and G B is perpendicular to B A. For the same reason, G B is perpendicular to B C. Because the straight line G B stands at right angles to the two straight lines B A and B C meeting one another in B, G B is perpendicular (XI. 4) to the plane of B A and B C. But it is perpendicular (*Const.*) to the plane of D E and E F. Therefore B G is perpendicular to each of the planes of A B and B C, and of D E and E F. But planes to which the same straight line is perpendicular, are parallel (XI. 14). Therefore the plane of A B and B C is parallel to the plane of D E and E F. Wherefore, if two straight lines, &c. Q. E. D.

PROP. XVI. THEOREM.

If two parallel planes be cut by another plane, their common sections with it are parallels.

Let the parallel planes A B and C D be cut by the plane F G, and let *their common sections* with it be E F and G H. E F is parallel to G H.

For, if they are not parallel, E F and G H shall meet if produced either on the side of F H or E G. Let them be produced on the side of F H, and meet in the point K.

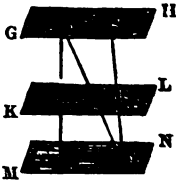


Because the straight line E F K is in the plane A B (XI. 1). Therefore the point K is in the plane A B. For the same reason, the point K is also in the plane C D. Wherefore the planes A B and C D being produced, meet one another. But they do not meet, because they are parallel (*Hyp.*). Therefore the straight lines E F and G H, do not meet when produced on the side of F H. In the same manner, it may be proved, that E F and G H do not meet when produced on the side of E G. But straight lines which are in the same plane, and do not meet, though produced either way (I. *Def.* 35), are parallel. Therefore E F is parallel to G H. Wherefore, if two parallel planes, &c. Q. E. D.

PROP. XVII. THEOREM.

If two straight lines be cut by parallel planes, they are cut in the same ratio.

Let the straight lines A B and C D be cut by the parallel planes G H, K L, and M N in the points A, E, and B; C, F, and D. As A E is to E B, so is C F to F D. Join A C, B D, and A D; and let A D meet the plane K L in the point X. Join E X and X F.



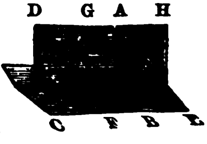
Because the two parallel planes K L and M N are cut by the plane B X, the common sections E X and B D are (XI. 16) parallel. Because the two parallel planes G H and K L are cut by the plane X C, the common sections A C and X F are parallel. Because E X is parallel to B D, a side of the triangle A B D; as A E to E B, so is A X to X D (XI. 2). Again, because X F is parallel to A C, a side of the triangle A D C; as A X is to X D, so is C F to F D. But it was proved that A X is to X D, as A E is to E B. Therefore (V. 11) as A E is to E B, so is C F to F D. Wherefore, if two straight lines, &c. Q. E. D.

PROP. XVIII. THEOREM.

If a straight line be at right angles to a plane, every plane which passes through it, is at right angles to that plane.

Let the straight line A B be at right angles to the plane C K. Every plane which passes through A B is at right angles to the plane C K.

Let any plane D E pass through A B, and let C E be the common section of the planes D E and C K. Take any point F in C E, and from F draw F G in the plane D E at right (I. 11) angles to C E.



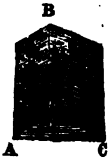
Because A B is perpendicular to the plane C K (XI. *Def.* 3), it is perpendicular to C E meeting it in that plane. Therefore A B F is a right angle. But G F B is likewise (*Const.*) a right angle. Therefore A B is parallel (I. 28) to F G. But A B is at right angles to the plane C K. Therefore F G is also (XI. 8) at right angles to the same plane. Because one plane is at right angles to another plane when the straight lines drawn in one of the planes, at right angles to their common section, are also at right angles (XI. *Def.* 4) to the other plane; and any straight line F G in the plane D E, which is at right angles to C E, the common section of the planes, has been proved to be perpendicular to the other plane C K. Therefore the plane D E is at right angles to the plane C K. In like manner, it may be proved that all planes which pass through A B are at right angles to the plane C K. Therefore, if a straight line, &c. Q. E. D.

PROP. XIX. THEOREM.

If two planes which cut one another be each of them perpendicular to a third plane; their common section is perpendicular to the same plane.

Let the two planes A B and B C be each of them perpendicular to a third plane. B D, the common section of the first two, is perpendicular to the third plane.

If it be not, from the point D (I. 11) draw, in the plane A B, the straight line D E at right angles to A D the common section of the plane A B with the third plane; and in the plane B C draw D F at right angles to C D the common section of the plane B C with the third plane.



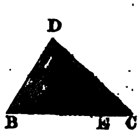
Because the plane A B is perpendicular to the third plane, and D E is drawn in the plane A B at right angles to A D, their common section, D E is perpendicular (XI. *Def.* 4) to the third plane. In the same manner, it may be proved, that D F is perpendicular to the third plane. But, from the point D two straight lines are drawn at right angles to the third plane, upon the same side of it, which is (XI. 13) impossible. Therefore, from the point D there cannot be any straight line at right angles to the third plane, except B D, the common section of the planes A B and B C. Wherefore B D is perpendicular to the third plane. Therefore, if two planes, &c. Q. E. D.

PROP. XX. THEOREM.

If a solid angle be contained by three plane angles, any two of them are greater than the third.

Let the solid angle at A be contained by the three plane angles B A C, C A D, and D A B. Any two of them are greater than the third.

If the angles B A C, C A D and D A B be all equal, it is evident, that any two of them are greater than the third. But if they are not, let B A C be that angle which is not less than either of the other two, and is greater than one of them D A B. At the point A in the straight line A B, make, in the plane of B A and A C, the angle B A E equal (I. 23) to the angle D A B. Make A E equal to A D, and through E draw B C cutting A B and A C in the points B and C. Join D B and D C.

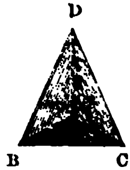


Because D A is equal to A E, and A B is common, the two D A and A B are equal to the two E A and A B, each to each; and the angle D A B is equal to the angle E A B. Therefore the base D B is equal (I. 4) to the base B E. Because B D and D C are greater (I. 20) than C B, and the one B D has been proved equal to B E a part of C B, therefore the other D C is greater (I. *Ax.* 5) than the remaining part E C. Because D A is equal to A E, and A C common, but the base D C is greater than the base E C. Therefore the angle D A C is greater (I. 25) than the angle E A C. But the angle D A B is equal (*Const.*) to the angle B A E. Therefore the angles D A B and D A C are together greater (I. *Ax.* 4) than the angles B A E and E A C, that is, than the angle B A C. But B A C is is not less than either of the angles D A B and D A C. Therefore B A C with either of them is greater than the other. Wherefore, if a solid angle, &c. Q. E. D.

PROP. XXI. THEOREM.

The plane angles by which every solid angle is contained, are together less than four right angles.

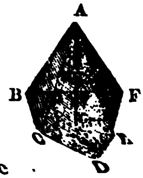
First, let the solid angle at A be contained by three plane angles B A C, C A D, and D A B. These three together are less than four right angles. Take in each of the straight lines A B, A C, and A D, any points B, C and D; and join B C, C D, and D B.



Because the solid angle at B is contained by the three plane angles C B A, A B D, and D B C, any two of them are greater (XI. 20) than the third. Therefore the angles C B A and A B D are greater than the angle D B C. For the same reason, the angles B C A and A C D are greater than the angle D C B; and the angles C D A and A D B greater than the angle B D C. Therefore the six angles C B A, A B D, B C A, A C D, C D A, and A D B are greater than the three angles D B C, B C D, and C D B. But the angles D B C, B C D, and C D B are equal to two right angles (I. 32), therefore the six angles C B A, A B D, B C A, A C D, C D A and A D B are greater than two right angles. Because the three angles of each of the triangles A B C, A C D and A D B are equal to two right angles. Therefore the nine angles of these three triangles, viz. the angles C B A, B A C, A C B, A C D, C D A, D A C, A D B, D B A, and B A D, are equal to six right angles. But of these, the six angles C B A, A C B, A C D, C D A, A D B, and D B A are greater than two right angles. Therefore the remaining three angles B A C, C A D, and D A B, which contain the solid angle at A, are less than four right angles.

Next, let the solid angle at A be contained by any number of plane angles B A C, C A D, D A E, E A F, and F A B. These angles are together less than four right angles.

Let the plane angles which form the solid angle be cut by a plane at any distance from its vertex, and let the common sections of it with those plane angles be B C, C D, D E, E F, and F B.



Because the solid angle at B is contained by three plane angles C B A, A B F, and F B C, of which any two are greater (XI. 20) than the third, the angles C B A and A B F, are greater than the angle F B C. For the same reason, the two plane angles at each of the points C, D, E, and F, viz., those angles which are at the bases of the triangles having the common vertex A, are greater than the third angle at the same point, which is one of the angles of the polygon *B C D E F*. Therefore all the angles at the bases of the triangles are together greater than all the angles of the polygon. Because all the angles of the triangles are together equal to twice as many right angles (I. 32) as there are triangles; that is, as there are sides in the polygon B C D E F; and all the angles of the polygon, together with four right angles, are likewise equal to twice as many right angles (I. 32, *Cor.* 1) as there are sides in the polygon. Therefore all the angles of the triangles are equal (I. *Ax.* 1) to all the angles of the polygon together with four right angles. But all the angles at the bases of the triangles are greater than all the angles of the polygon, as has been proved. Therefore the remaining angles of the triangles, viz., those at the vertex, which contain the solid angle at A, are less than four right angles. Therefore, every solid angle, &c. Q. E. D.

This demonstration holds good only when the solid angles are convex; and no other solid angle is contemplated by Euclid.

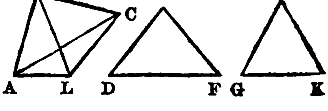
Corollary.—There cannot be more than *five regular polyhedrons.* For each of the angles of a polyhedron must, by this proposition, be *less than four* right angles. But *six* angles of an equilateral triangle, *four* angles of a square, and *three* angles of a hexagon, are respectively equal to *four* right angles. And, *four* angles of a pentagon, and *three* angles of all *regular* polygons, of a greater number of sides than a hexagon, are respectively equal to *more than four* right angles. Therefore, *three*, *four*, or *five* equilateral triangles, *three* squares and *three* pentagons, are the only *regular* plane figures of which a regular polyhedron can be constructed. Wherefore there cannot be more than *five regular polydedrons.*

PROP. XXII. THEOREM.

If every two of three plane angles be greater than the third, and if the straight lines which contain them be all equal; a triangle may be made of the straight lines that join the extremities of those equal straight lines.

Let A B C, D E F, and G H K be the three plane angles, every two of which are greater than the third, and let the straight lines A B, B C, D E, E F, G H, and H K be all equal. If A C, D F, and G K be joined, a triangle may be made of three straight lines equal to A C, D F, and G K; that is, every two of them are together greater than the third.

If the angles at B, E, and H are equal, the straight lines A C, D F, and G K are also equal (I. 4), and any two of them are greater than the third. But if the angles are not all equal, let the angle A B C be not less than either of the two angles D E F and G H K. Since the straight line A C is not less (I. 4, or 24) than either of the other two D F and G K; it is plain, that A C, together with either of the other two, must be greater than the third. Also, D F with G K is greater than A C. For, at the point B in the straight line A B make (I. 23) the angle A B L equal to the angle G H K, and make B L equal to A B. Join A L and L C.



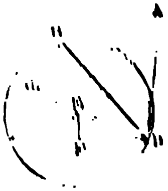
Because A B and B L are equal to G H and H K, each to each, and the angle A B L to the angle G H K, the base A L is equal (I. 4) to the base G K. Because the *angles at E and H* are together greater (*Hyp.*) than the angle A B C, *of which the angle* at H is equal to A B L. Therefore the remaining *angle at E is greater* (I. *Ax.* 5) than the angle L B C. Because the two sides LB and B C are equal to the two D E and E F, each to each, and the angle D E F is greater than the angle L B C, the base D F is greater (I. 24) than the base L C. But it has been proved that G K is equal to A L. Therefore D F and G K are greater (I. *Ax.* 4) than A L and L C. But A L and L C are greater (I. 20) than A C. Much more therefore are D F and G K greater than A C. Wherefore every two of these straight nes A C, D F, and G K are greater than the third; and a triangle may e made (I. 22), the sides of which shall be equal to A C, D F and K. Therefore, if every two, &c. Q. E. D.

The following lemma is introduced here for the purpose of shortening the demonstration of the twenty-third proposition, which follows it.

PROP. A.* LEMMA.

If on the edges of a solid angle contained by three plane angles, that is, on the common sections of those planes, points be taken at equal distances from the vertex, and a triangle be formed by joining those points; either of these distances is greater than the radius of the circle described about the triangle.

Let the solid angle at A be contained by three plane angles B A C, A D, and D A B; let the points B, C, and D be taken at equal distances om the vertex A; and let the triangle B C D be formed by joining ese points. The distance A B, A C, or A D is reater than the radius E C or E D of the circle C D described about the triangle B C D.



Bisect any two sides B C and C D of the triangle C D, which do not pass through the centre E; and oin E A, E F, E G, A F, and A G.

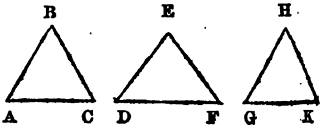
Because the triangles C A D and C E D are osceles (*Hyp.*, and I. *Def.* 15), and A F and E F re drawn from their vertices bisecting the common ase C D, the angles A F C and E F C are right ngles. Therefore C F is perpendicular to the plane A F E. But the plane B C D passes through C F. Therefore the plane B C D is perpendicular to the lane A E F, and A E F to B C D. In the same manner, it may be roved that the plane A E G is perpendicular to the plane B C D. Therefore A E, the common section of the planes A E F and A E G, is perpendicular to the plane B C D (XI. 19). Therefore A E C is a right ngle, and E A C an acute angle (I. 32). Wherefore A C is greater than E C (I. 19). Therefore, if on the edges, &c. Q. E. D.

Corollary.—If there be three plane angles such that a solid angle can be contained by them; and if the legs of these angles be made all equal to each other, and three straight lines be drawn joining their extremities; the length of each leg is greater than the radius of the circle described about a triangle whose sides are equal to those three straight lines.

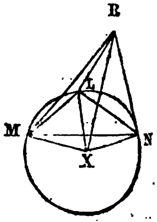
PROP. XXIII. PROBLEM.

To make a solid angle which shall be contained by three given plane angles, any two of them being greater (XI. 20) *than the third, and all three together* (XI. 21) *less than four right angles.*

Let the three given plane angles be A B C, D E F, G H K, any two of which are greater than the third, and all of them together less than four right angles. It is required to make a solid angle contained by three plane angles equal to A B C, D E F and G H K, each to each.



From the straight lines which contain the angles cut off A B, B C, D E, E F, G H, and H K, all equal to one another; and join A C, D F, and G K. Describe a triangle (XI. 22) L M N, whose three sides L M, M N, and N L are equal to A C, D F and G K (I. 22), so that L M is equal to A C, M N to D F, and L N to G K. About the triangle L M N describe (IV. 5) a circle, and find (III. 1) its centre X. From the point X, draw (XI. 12) X R at right angles to the plane of the circle L M N. Because A B is greater than L X (XI. A*), find a square (II. 14) equal to the excess of the square of A B above the square of L X, and make R X equal to its side, and join R L, R M, and R N. The solid angle at R is the angle required.



Because R X is perpendicular to the plane of the circle L M N, it is (XI. *Def.* 3) perpendicular to each of the straight lines L X, M X, N X. And because L X is equal to M X, and X R common, and at right angles to each of them, the base R L is equal (I. 4.) to the base R M. For the same reason, R N is equal to each of the two R L and R M: therefore the three straight lines R L, R M, and R N, are all equal. And because the square of X R is equal to the excess of the square of A B above the square of L X (*Const.*). Therefore the square of A B is equal to the squares of L X and X R. But the square of R L is equal (I. 47) to the same squares, because L X R is a right angle. Therefore the square of A B is equal to the square of R L, and the straight line A B to the straight line R L. But each of the straight lines B C, D E, E F, G H, and H K, is equal to A B, and each of the two R M and R N is equal to R L; therefore A B, B C, D E, E F, G H, and H K are each of them equal to each of the straight lines R L, R M, and R N. Because R L and R M are equal to A B and B C, each to each, and the base L M to the base A C; the angle L R M is equal (I. 8) to the angle A B C. For the same reason, the angle M R N is equal to the angle D E F, and the angle N R L to the angle G H K. Therefore the solid angle at R, is contained by three plane angles L R M, M R N, and N R L, which are equal to the three given plane angles A B C, D E F, and G H K, each to each. Which was to be done.

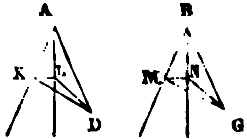
The following propositions were introduced by Dr. Simson into this book, for the purpose of establishing the subsequent propositions on a firm basis.

PROP. A. THEOREM.

If two solid angles be each contained by three plane angles, which are equal to one another, each to each; the planes in which the equal angles are, have the same inclination to one another.

Let there be two solid angles at the points A and B; let the angle at *A be contained by* the three plane angles C A D, C A E, and E A D; and the angle at B by the three plane angles F B G, F B H and H B G; and let the angle C A D be equal to the angle F B G, the angle C A E to the angle F B H, and the angle E A D to the angle H B G. The planes in which the equal angles are have the same inclination to one another.

In the straight line A C take any point K, and from K (I. 11) draw in the plane C A D the straight line K D at right angles to A C, and in the plane C A E the straight line K L at right angles to A C; the angle D K L is the inclination (XI. *Def.* 6) of the plane C A D to the plane C A E. In B F take B M equal to A K, and from the point M draw in the planes F B G and F B H, the straight lines M G and M N at right angles to B F; the angle G M N is the inclination (XI. *Def.* 6) of the plane F B G to the plane F B H. Join L D, and N G.

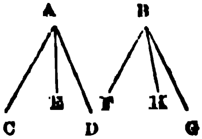


Because in the triangles K A D and M B G, the angles K A D and M B G are (*Hyp.*) equal, as also the right angles A K D and B M G, and the sides A K and B M, adjacent to the equal angles, are equal to one another. Therefore K D is equal (I. 26) to M G, and A D to B G. For the same reason, in the triangles K A L and M B N, K L is equal to M N, and A L to B N. Therefore in the triangles L A D and N B G, L A and A D are equal to N B and B G, each to each; and they contain equal angles. Therefore the base K D is equal (I. 4) to the base M G. Because, in the triangles K L D and M N G, the sides D K and K L are equal to G M and M N, each to each; and the base L D to the base N G. Therefore the angle D K L is equal to (I. 8) the angle G M N. But the angle D K L is the inclination of the plane C A D to the plane C A E, and the angle G M N is the inclination of the plane F B G to the plane F B H. Therefore these planes have the same inclination (XI. *Def.* 7) to one another. In the same manner, it may be demonstrated, that the other planes in which the equal angles are, have the same inclination to one another. Therefore, if two solid angles, &c. Q. E. D.

PROP. B. THEOREM.

If two solid angles be each contained by three plane angles which are equal to one another, each to each, and similarly situated; these solid angles are equal to one another.

Let there be two solid angles at A and B, of which the solid angle at A is contained by the three plane angles C A D, C A E, and E A D; and the solid angle at B, by the three plane angles F B G, F B H, and H B G, and let C A D be equal to F B G; C A E to F B H; and E A D to H B G. The solid angle at A is equal to the solid angle at B.

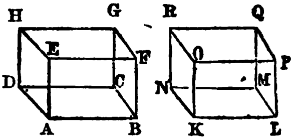


Let the solid angle at A be applied to the solid angle at B. The plane angle C A D being applied to the plane angle F B G, so that the point A may coincide with the point B, and the straight line A C with B F; *the straight line* A D shall coincide *with the straight line* B G, because the angle C A D is equal to the angle *F B G*. But the inclination of the plane C A E to the plane C A D is equal (XI. *A.*) to the inclination of the plane F B H to the plane F B G. Therefore the plane C A E shall coincide with the plane F B H, because the planes C A D and F B G coincide with one another. Because the straight lines A C and B F coincide, and the angle C A E is equal to the angle F B H. Therefore the straight line A E shall coincide with the straight line B H. But A D coincides with B G. Therefore the plane E A D coincides with the plane H B G. Wherefore the solid angle at A coincides with the solid angle at B, and they are equal (I. *Ax.* 8) to one another. Therefore if two solid, &c. Q. E. D.

PROP. C. THEOREM.

Solid figures which are contained by the same number of equal and similar planes similarly situated, and having none of their solid angles contained by more than three plane angles, are equal and similar to one another.

Let A G and K Q be two solid figures contained by the same number of similar and equal planes, similarly situated, viz., the plane A C similar and equal to the plane K M; the plane A F to the plane K P; B G to L Q; G D to Q N; D E to N O; and, F H similar and equal to P R. The solid figure A G is equal and similar to the solid figure K Q.



Because the solid angle at A is contained by the three plane angles B A D, B A E, and E A D, and the solid angle at K by the three plane angles L K N, L K O, and O K N, which are equal to them, each to each. Therefore the solid angle at A is equal (XI. B) to the solid angle at K. In the same manner, it may be shown, that the other solid angles of the figures are equal to one another. Let the solid figure A G be applied to the solid figure K Q. The plane figure A C being applied to the plane figure K M, so that the straight line A B may coincide with K L, the figure A C must coincide with the figure K M, because they are equal and similar. Therefore the straight lines A D, D C, and C B coincide with the straight lines K N, N M, and M L, each with each; and the points A, D, C, and B with the points K, N, M, and L; and the solid angle at A coincides with (XI. B) the solid angle at K. But the plane A F coincides with the plane K P, and the figure A F with the figure K P, because they are equal and similar to one another. Therefore the straight lines A E, E F, and F B coincide with K O, O P, and P L; and the points E and F with the points O and P. In the same manner, the figure A H coincides with the figure K R, and the straight line D H with N R, and the point H with the point R. Because the solid angle at B is equal to the solid angle at L, it may be proved, in the same manner, that the figure B G coincides with the figure L Q, and the straight line C G with M Q, and the point G with the point Q. Therefore, all the planes and sides of the solid figure A G coincide with the planes and sides of the solid figure K Q, and A G is equal and similar to K Q. In the same manner, any other solid figures whatever contained by the same number of equal and similar planes, similarly situated, and having none of their solid angles contained by more than three plane angles, may be proved to be equal and similar to one another. Q. E. D.

This proposition is not sufficiently general, as it refers only to rectangular parallelopipeds, which are equal *by coincidence*. To extend it to those solids which are equal *by symmetry*, would take more room than we can spare; and would be too great an encroachment on Euclid's Elements.

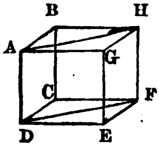
Corollary.—All regular polyhedrons are similar solids, and if they have one edge equal, they are every way equal.

PROP. XXIV. THEOREM.

If a solid be contained by six planes, two and two of which are parallel, the opposite planes are similar and equal parallelograms.

Let the solid C G be contained by the parallel planes A C and G F; B G and C E; F B and A E. Its opposite planes are similar and equal parallelograms.

Because the two parallel planes B G and C E are cut by the plane A C, their common sections A B and C D (XI. 16) are parallel. Because the two parallel planes B F and A E are cut by the plane A C, their common sections A D and B C (XI. 16) are parallel. But A B is parallel to C D. Therefore A C is a parallelogram. In like manner, it may be proved that each of the figures C E, F G, G B, B F, and A E is a parallelogram. Join A H and D F.



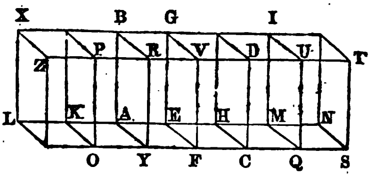
Because A B is parallel to D C, and B H to C F; the two straight lines A B and B H, which meet one another, are parallel to the two straight lines D C and C F, which meet one another, and are not in the same plane with the other two. Therefore they contain (XI. 10) equal angles; that is, the angle A B H is equal to the angle D C F. Because A B and B H are equal to D C and C F, each to each, and the angle A B H is equal to the angle D C F. Therefore the base A H is equal (I. 4) to the base D F, and the triangle A B H to the triangle D C F. But the parallelogram B G is double (I. 34) of the triangle A B H, and the parallelogram C E double of the triangle D C F. Therefore the parallelogram B G is equal and similar to the parallelogram C E. In the same manner it may be proved that the parallelogram A C is equal and similar to the parallelogram G F, and the parallelogram A E to the parallelogram B F. Therefore if a solid, &c. Q. E. D.

PROP. XXV. THEOREM.

If a parallelopiped be cut by a plane parallel to two of its opposite planes, it is divided into two solids or parallelopipeds, which are to one another as their bases.

Let the parallelopiped B C be cut by the plane E V, parallel to the opposite planes A R and H D, and dividing it into the two solids A V and E D. The base A F of the one is to the base H F of the other, as the solid A V is to the solid E D.

Produce A H both ways, and take any number of straight lines H M and M N, each equal to E H, and any number A K and K L, each equal to E A, and complete the parallelograms L O, K Y, H Q, and M S, and the solids L P, K R, H U, and M T.



Because the straight lines LK, KA, and AE are all equal, the parallelograms LO, KY, and AF are (I. 36) all equal; and likewise the parallelograms KX, KB, and AG. Also (XI. 24) the parallelograms LZ, KP, and AR are equal, because they are opposite planes. For the same reason, the parallelograms EC, HQ, and MS are equal (I. 36), and the parallelograms HG, HI, and IN: as also (XI. 24) HD, MU, and NT. Therefore three planes of the solid LP are equal and similar to three planes of the solid KR, as also to three planes of the solid AV. But the three planes opposite to these three are equal and similar (XI. 24) to them in the several solids, and none of their solid angles are contained by more than three plane angles; therefore the three solids LP, KR, and AV are equal (XI. C) to one another. For the same reason, the three solids ED, HU, and MT are equal to one another. Therefore, what multiple soever the base LF is of the base AF, the same multiple is the solid LV of the solid AV; and whatever multiple the base NF is of the base HF, the same multiple is the solid NV of the solid ED: and if the base LF be equal to the base NF, the solid LV is equal (XI. C) to the solid NV; if the base LF be greater than the base NF, the solid LV is greater than the solid NV; and if less, less. Because there are four magnitudes, viz., the two bases AF and FH, and the two solids AV and ED; and of the base AF and the solid AV, the base LF and the solid LV are any equimultiples whatever; and of the base FH and the solid ED, the base FN and the solid NV are any equimultiples whatever: and it has been proved, that if the base LF is greater than the base FN, the solid LV is greater than the solid NV: if equal, equal; and if less, less. Therefore (V. *Def.* 5) the base AF is to the base FH, as the solid AV is to the solid ED. Wherefore, if a parallelopiped, &c. Q. E. D.

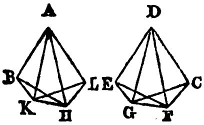
PROP. XXVI. PROBLEM.

At a given point in a given straight line, to make a solid angle equal to a given solid angle contained by three plane angles.

Let AB be a given straight line, A a given point in it, and D a given solid angle contained by the three plane angles EDC, EDF and FDC. It is required to make at the point A in the straight line AB a solid angle equal to the solid angle D.

In the straight line DF take any point F, and from it draw (XI. 11) FG perpendicular to the plane EDC, meeting that plane in G, and join DG. At the point A, in the straight line AB, make (I. 23) the angle BAL equal to the angle EDC, and in the plane BAL make the angle BAK equal to the angle EDG. Make AK equal to DG, and from the point K draw (XI. 12) KH at right angles, to the plane BAL. Make KH equal to GF, and join AH. The solid angle at A which is contained by the three plane angles BAL, BAH and HAL is equal to the solid angle at D contained by the three plane angles EDC, EDF, and FDC. Take the equal straight lines AB and DE, and join HB, KB, FE, and GE.

Because FG is perpendicular to the plane EDC, each of the angles FGD and FGE (XI. *Def.* 3) is a right angle. For the same reason, HKA and HKB are right angles. Because KA and AB are equal to GD and DE, each to each, and they contain equal angles. Therefore the base BK is equal (I. 4) to the base EG. But KH is equal (*Const.*) to GF and HKB and FGE are right angles. Therefore HB is equal (I. 4) to FE. Because AK and KH are equal to DG and GF, each to each, and contain right angles, the base AH is equal to the base DF. But AB is equal to DE (*Const.*). Therefore HA and AB are equal to FD and DE, each to each; and the base HB is equal to the base FE Therefore the angle BAH is equal (I. 8) to the angle EDF. Again, make AL and DC equal, and join KL, HL, GC, and FC. Because the whole angle BAL is equal to the whole EDC, and the parts of them BAK and EDG are (*Const.*) equal. Therefore the remaining angle KAL is equal to the remaining angle GDC. Because KA and AL are equal to GD and DC, each to each, and they contain equal angles, the base KL is equal (I. 4) to the base GC. But KH is equal to GF (*Const.*). Therefore LK and KH are equal to CG and GF, each to each; and they contain right (XI. *Def.* 3) angles. Therefore the base HL is equal (I. 4) to the base FC. Because HA and AL are equal to FD and DC, each to each, and the base HL to the base FC, the angle HAL is equal (I. 8) to the angle FDC. Because the three plane angles BAL, BAH, and HAL, which contain the solid angle at A, are equal to the three plane angles EDC, EDF, and FDC, which contain the solid angle at D, each to each, and are situated in the same order, the solid angle at A is equal (XI. B) to the solid angle at D. Therefore at a given point in a given straight line a solid angle has been made equal to a given solid angle contained by three plane angles. Q. E. F.



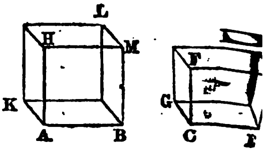
PROP. XXVII. PROBLEM.

To describe upon a given straight line, as one of its edges, a parallelopiped similar and similarly situated to a given parallelopiped.

Let AB be the given straight line, and CD the given parallelopiped. It is required upon AB to describe a parallelopiped similar and similarly situated to CD.

At the point A in AB make (I. 26) a solid angle equal to the solid angle at C, and let BAK, KAH, and HAB be the three plane angles which contain it, so that BAK is equal to ECG, KAH to GCF, and HAB to FCE. As EC is to CG, so make (VI. 12) BA to AK; and as GC is to CF, so make (VI. 12) KA to AH; wherefore, *ex æquali* (V. 22), as EC is to CF, so is BA to AH. Complete the parallelogram BH, and the solid AL. The parallelopiped AL shall be similar and similarly situated to the parallelopiped CD.

Because EC is to GC, as BA is to AK, the sides about the equal angles ECG and BAK are proportionals. Therefore the parallelogram BK is similar (VI. *Def.* 1) to the parallelogram EG. For the same reason, the parallelogram K H is similar to the parallelogram G F, a H B to F E. Therefore three parallelograms of the solid A L are similar to three of the solid C D: and the three opposite ones in each solid are equal (XI. 24) and similar to these, each to each. Also, because the plane angles which contain the solid angles of the figures are equal, each to each, and are situated in the same order, the solid angles are equal (XI. B), each to each. Therefore the solid A L is similar (XI. *Def.* 11) to the solid C D. Wherefore from a given straight *line* A B a parallelopiped A L has been described similar and similarly situated to the given parallelopiped C D. Q. E. F.

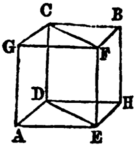


PROP. XXVIII. THEOREM.

If a parallelopiped be cut by a plane passing through the diagonals of two of its opposite planes; it is cut into two equal triangular prisms.

Let A B be a parallelopiped, and D E and C F the diagonals of the opposite parallelograms A H and G B, viz., those which are drawn between the equal angles in each.

Because C D and F E are each of them parallel to G A, and not in the same plane with it, C D and F E are (XI. 9) parallel. Therefore the diagonals C F and D E, are in the plane in which the parallels are, and are themselves (XI. 16) parallels. The plane C E cuts the solid A B into two equal triangular prisms.



Because the triangle C G F is equal (I. 34) to the triangle C B F, and the triangle D A E to the triangle D H E; and the parallelogram C A is equal (XI. 24) and similar to the opposite parallelogram B E, and the parallelogram G E to the parallelogram C H. The prism contained by the two triangles C G F and D A E, and the three parallelograms C A, G E, and E C is equal (XI. C) to the prism contained by the two triangles C B F and D H E, and the three parallelograms B E, C H and E C; because they are contained by the same number of equal and similar planes, similarly situated, and none of their solid angles are contained by more than three plane angles. Therefore the solid A B is cut into two equal triangular prisms by the plane C E. Q. E. D.

This demonstration is not sufficiently general, as it ought to include *oblique* as well as *right* parallelopipeds. The prisms of *oblique* parallelopipeds, as in the next proposition, are only equal *by symmetry*. See Legendre's "Geometry."

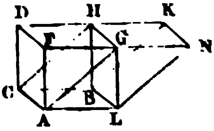
"N.B.—The insisting straight lines of a parallelopiped, mentioned in the next and some following propositions, are the sides of the parallelograms betwixt the base and the opposite plane parallel to it."

PROP. XXIX. THEOREM.

Parallelopipeds upon the same base, and of the same altitude, the insisting straight lines of which are terminated in the same straight lines in the plane opposite to the base, are equal to one another.

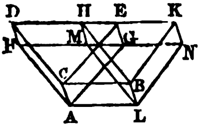
Let the parallelopipeds A H and A K be upon the same base A B, and of the same altitude, and let their insisting straight lines A F, A G, L M, and L N be terminated in the same straight line F N; and C D, C E, B H, and B K be terminated in the same straight line D K. The solid A H is equal to the solid A K.

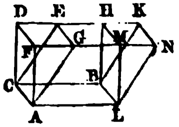
First, let the parallelograms D G and H N, which are opposite to the base A B, have a common side H G.



Because the solid F B is bisected by the plane A H passing through the diagonals A G and C H, of the opposite planes L F and B D (XI. 28), the solid F B is double of the prism A B G. Because the solid A K is bisected by the plane L H passing through the diagonals L G and B H of the opposite planes A N and C K, the solid A K is double of the same prism A B G. Therefore the solid F B is equal (I. *Ax.* 6) to the solid A K.

Next, let the parallelograms D M and E N, which are opposite to the base A B, have no common side.





Because C H and C K are parallelograms, C B is equal (I. 34) to each of the opposite sides D H and E K. Therefore D H is equal to E K. Add or take away the common part H E; and D E is equal (I. *Ax.* 2 or 3) to H K. Therefore the triangle C D E is equal (I. 38) to the triangle B H K, and the parallelogram D G is equal (I. 36) to the parallelogram H N. For the same reason, the triangle A F G is equal to the triangle L M N. Also the parallelogram C F is equal (XI. 24) to the parallelogram B M, and C G to B N. Therefore the prism which is contained by the two triangles A F G and C D E, and the three parallelograms A D, D G and G C, is equal (XI. C) to the prism contained by the two triangles L M N and B H K, and the three parallelograms B M, M K and K L. If therefore the prism N H L be taken from the whole solid A B N D, the remainder is the solid A H; and if from the same solid the prism G D A be taken the remainder is the solid A K. Therefore the parallelopiped A H is equal (I. *Ax.* 3) to the parallelopiped A K. Therefore parallelopipeds, &c. Q. E. D.

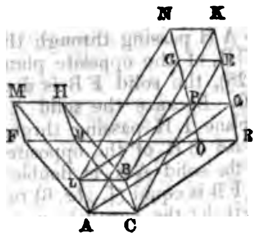
PROP. XXX. THEOREM.

Parallelopipeds upon the the same base, and of the same altitude, the insisting straight lines of which are not terminated in the same straight lines in the plane opposite to the base, are equal to one another.

Let the parallelopipeds C M and C N be upon the same base A B, and of the same altitude, but their insisting straight lines A F, A G, L M, L N, C D, C E, B H, and B K not terminated in the same straight lines. The solids C M and C N are equal to one another.

Produce F D and M H, also N G and K E, and let them meet one another in the points O, P, Q, and R; and join A O, L P, B Q, and C R.

Because the plane L H is parallel to the opposite plane A D, and is that in which are the parallels L B and M Q, and the figure B P; and the plane A D is that in which are the parallels A C and F R, and the figure C O. Therefore the figures B P and C O are in parallel planes. Because the plane A N is parallel to the opposite plane C K, and is that in which are the parallels A L, and O N, and the figure A P; and the plane C K is that in which are the parallels C B and R K, and the figure C Q. Therefore the figures A P and C Q are in parallel planes. But the planes A B and O Q are (*Hyp.*) parallel. Therefore the solid C P is a parallelopiped. But the solid C M is equal (XI. 29) to the solid C P, because they are upon the same base A B, and their insisting straight lines A F, A O, C D, and C R; L M, L P, B H, and B Q are terminated in the same straight lines F R and M Q; and the solid C P is equal (XI. 29) to the solid C N, for they are upon the same base A B, and their insisting straight lines A O, A G, L P, and L N; C R, C E, B Q, and B K are terminated in the same straight lines O N and R K. Therefore the solid C M is equal to the solid C N. Wherefore parallelopipeds, &c. Q. E. D.

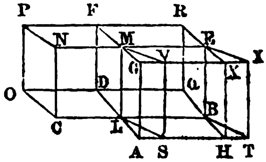


PROP. XXXI. THEOREM.

Parallelopipeds, which are upon equal bases, and of the same altitude, are equal to one another.

Let the solid parallelopipeds A E and C F be upon equal bases A B and C D, and of the same altitude. The solid A E is equal to the solid C F.

First, let the insisting straight lines be at right angles to the bases A B and C D, and let the bases be placed in the same plane, so that the sides C L and L B may be in a straight line. The straight line L M, which is at right angles to the plane in which the bases are at the point L, is common (XI. 13) to the two solids A E and C F. Let the other insisting lines of the solids be A G, H K, and B E; D F, O P, and C N. Let the angle A L B, in the first case, be equal to the angle C L D. The other sides A L and L D are in a straight line (I. 14). Produce O D and H B; let them meet in Q; and complete the solid parallelopiped L R, of which the base is the parallelogram L Q, and L M is one of its insisting straight lines.



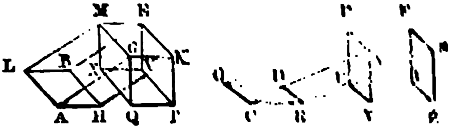
Because A B is equal to C D (*Hyp.*). Therefore A B is to L Q (V. 7), as C D is to L Q. Because the parallelopiped A R is cut by the plane L E, parallel to the opposite planes A K and D R. Therefore as the base A B is to the base L Q, so is (XI. 25) the solid A E to the solid L R. Because the parallelopiped C R is cut by the plane L F parallel to the opposite planes C P and B R. Therefore as the base C D is to the base L Q, so is he solid C F to the solid L R. But the base A B is to the base L Q, as he base C D is to the base L Q, as before proved. Therefore, as the solid .E is to the solid L R (V. 11), so is the solid C F to the solid L R. herefore the solid A E is equal (V. 9) to the solid C F.

In the second case, let the angles S L B and C L D be unequal. The lid S E is equal to the solid C F.

Produce D L and T S until they meet in A, and from B draw B H rallel to D A; and let H B and O D produced meet in Q, and complete e solids A E and L R.

The solid A E is equal (XI. 29) to the solid S E, because they are upon e same base L E, and of the same altitude, and their insisting straight ies, viz. L A, L S, B H, and B T; M G, M V, E K, and E X, are in the me straight lines A T and G X. Because the parallelogram A B is equal .35) to S B, for they are upon the same base L B, and between the same rallels L B and A T; and the base S B is equal to the base C D (Hyp.). erefore the base A B is equal to the base C D; and the angle A L B is ual to the angle C L D. Wherefore, by the first case, the solid A E is ual to the solid C F; but it was shown that the solid A E is equal to e solid S E. Therefore the solid S E is equal to the solid C F.

Secondly, let the insisting straight lines A G, H K, B E, and L M; C N, S, D F, and O P be not right angles to the ses A B and C D; in is case likewise the lid A E is equal to the lid C F.



From the points G, K, and M, N, S, F, and P, aw the straight lines G Q, K T, E V, and M X; N Y, S Z, F I, and P U, rpendicular (XI. 11) to the planes in which are the bases A B and D; and let them meet these planes in the points Q, T, V, and X; Y, I, and U. Join Q T, T V, V X, and X Q; Y Z, Z I, I U, and U Y.

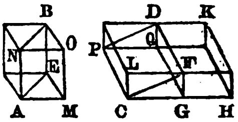
Because G Q and K T are at right angles to the same plane, they are rallel (XI. 6) to one another: and M G and E K are parallels. Therefore re the planes M Q and E T, of which one passes through M G and G Q, id the other through E K and K T, which are parallel to M G and G Q, id not in the same plane with them, are parallel (XI. 15) to one another. or the same reason, the planes M V and G T are parallel to one another. erefore the solid Q E is a parallelopiped. In like manner, it may be oved, that the solid Y F is a parallelopiped. But, from what has been monstrated, the solid E Q is equal to the solid F Y, because they are on equal bases M K and P S, and of the same altitude, and have their sisting straight lines at right angles to the bases. Again, the solid Q is equal (XI. 29 or 30) to the solid A E, and the solid F Y to the lid C F, because they are upon the same bases and of the same altitude. erefore the solid A E is equal to the solid C F. Wherefore, solid rallelopipeds, &c. Q. E. D.

This proposition is the foundation of the mensuration of solids. As the solid content of a *right* parallelopiped is found in practice by multiplying the area of its *base by its altitude*, so the solid content of every oblique parallelopiped is found *by multiplying* the area of its base by its altitude. And the same rule applies to *prisms*.

PROP. XXXII. THEOREM.

Parallelopipeds which have the same altitude are to one another as their bases.

Let A B and C D be parallelopipeds of the same altitude. They are to one another as their bases; that is, as the base M N is to the base C F so is the solid A B to the solid C D.



To the straight line F G apply the parallelogram F H equal (I. *Cor.* 45) to M N, so that the angle F G H may be equal to the angle L C G; and upon the base F H complete the parallelopiped G K, one of whose insisting straight lines is F D, so that the solids C D and G K are of the same altitude.

The solid A B is equal (XI. 31) to the solid G K, because they are upon equal bases M N and F H, and are of the same altitude. Because the parallelopiped C K is cut by the plane D G, which is parallel to its opposite planes, the base H F is (XI. 25) to the base L G, as the solid H D to the solid D C. But the base H F is equal to the base M N, and the solid G K to the solid A B. Therefore as the base M N is to the base C F, so is the solid A B to the solid C D. Wherefore, solid parallelopipeds, &c. Q. E. D.

COROLLARY.—From this it is manifest, that prisms upon triangular bases and of the same altitude, are to one another as their bases. Let the prisms, the bases of which are the triangles A E M and C F G, and N B O and P D Q the triangles opposite to them, have the same altitude: they are to one another as their bases.

Complete the parallelograms A E and C F, and the parallelopipeds A B and C D, in the first of which let M O, and in the other, let G Q be one of the insisting straight lines.

Because the parallelopipeds A B and C D have the same altitude, they are to one another as the base M N is to the base C F. Therefore the prisms, which are their (XI. 28) halves, are to one another, as their bases; that is, as the triangle A E M to the triangle C F G.

Corollary.—Parallelopipeds which have equal bases are to one another as their altitudes.

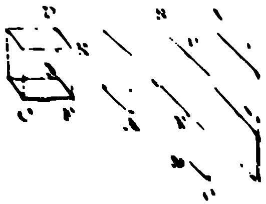
PROP. XXXIII. THEOREM.

Similar parallelopipeds are one to another in the triplicate ratio of their homologous sides or edges.

Let A B and C D be similar parallelopipeds, having the side A E homologous to the side C F. The solid A B has to the solid C D the triplicate ratio of that which A E has to C F.

Produce A E, G E, and H E, and in them take E K equal to C F, E L equal to F N, and E M equal to F R. Complete the parallelogram K L, and the solid K O.

Because the solids A B and C D are similar (*Hyp.*), the angle A E G is equal *to the angle C F N.* But the angle A E G is equal to the angle K E L *(I. 15). Therefore* the angle K E L is equal to the angle C F N. But K E *and E L* are equal to C F and F N (Const.). Therefore the parallelo-



gram KL is similar and equal to the parallelogram CN. For the same reason the parallelogram MK is similar and equal to the parallelogram CR: and also OE to FD. Therefore three parallelograms of the solid KO are equal and similar to three parallelograms of the solid CD. But the three opposite ones in each solid are equal (XI. 24) and similar to these. Therefore the solid KO is equal (XI. C) and similar to the solid CD. Complete the parallelogram GK: and upon the bases GK and KL complete the solids EX and LP, so that EH be an insisting straight line in each of them and that they are of the same altitude with the solid AB.

Because the solids AB and CD are similar, AE is to EG as CF is to FN. Therefore, by permutation, AE is to CF, as EG is to FN, and as EH is to FR. But FC is equal to EK, FN to EL, and FR to EM. Therefore AE is to EK, as EG is to EL, and as HE is to EM. But AE is to EK (VI. 1), as the parallelogram AG is to the parallelogram GK; and GE is to EL, as (VI. 1) GK is to KL, and HE is to EM (VI. 1), as PE is to KM. Therefore the parallelogram AG is to the parallelogram GK, as GK is to KL, and as PE is to KM. But AG is to GK, as (XI. 25) the solid AB is to the solid EX; and GK is to KL (XI. 25), as the solid EX is to the solid PL; and PE is to KM, as (XI. 25) the solid PL is to the solid KO. Therefore the solid AB is to the solid EX, as EX is to PL, and as PL is to KO. But if four magnitudes be continual proportionals, the first is said to have to the fourth the triplicate (V. *Def.* 11) ratio of that which it has to the second. Therefore the solid AB has to the solid KO, the triplicate ratio of that which AB has to EX. But AB is to EX, as the parallelogram AG is to the parallelogram GK, and as the straight line AE is to the straight line EK. Therefore the solid AB has to the solid KO the triplicate ratio of that which AE has to EK. But the solid KO is equal to the solid CD, and the straight line EK is equal to the straight line CF. Therefore the solid AB has to the solid CD the triplicate ratio of that which the side AE has to the homologous side CF. Therefore, similar parallelopipeds, &c. Q. E. D.

Corollary.—From this it is manifest that, if four straight lines be continual proportionals, as the first is to the fourth, so is the parallelepiped described from the first as a side or edge to the similar solid similarly described from the second as a side or edge, because the first straight line has to the fourth the triplicate ratio of that which it has to the second.

This proposition is one of the most important in this Book. It is the foundation of the rule for determining the solid content of all similar solids bounded by planes.

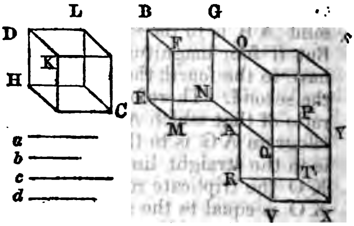
PROP. D. THEOREM.

Parallelopipeds which are contained by parallelograms equiangular to one another, each to each, that is, of which the solid angles are equal, each to each, have to one another the ratio which is the same with the ratio compounded of the ratios of their sides.

Let A B and C D be parallelopipeds, of which A B is contained by the parallelograms A E, A F, and A G, equiangular, each to each, to the parallelograms C H, C K, and C L, which contain the solid C D. The ratio which the solid A B has to the solid C D, is the same with that which is compounded of the ratios of the sides A M to D L, A N to D K, and A O to D H.

Produce M A, N A, and O A to P, Q, and R, making A P equal to D L, A Q to D K, and A R to D H. Complete the parallelopiped A X contained by the parallelograms A S, A T, and A V, similar and equal to C H, C K, and C L, each to each. The solid A X is equal (XI. C) to the solid C D. Complete likewise the solid A Y, of which the base is A S, and A O one of its insisting straight lines. Take any straight line *a*, and as M A is to A P, so make (VI. 12) *a* to *b*; and as N A is to A Q, so make *b* to *c*; and as A O is to A R, so make *c* to *d*.

Because the parallelogram A E is equiangular to A S, A E is to A S as *a* is to *c* (VI. 23). But the solids A B and A Y, between the parallel planes B O Y and E A S, are of the same altitude. Therefore the solid A B is to the solid A Y as (XI. 32) the base A E is to the base A S; that is, as *a* is to *c*. But the solid A Y is to the solid A X as (XI. 25) the base O Q is to the base Q R; that is, as O A is to A R; that is, as *c* is to *d*. Because the solid A B is to the solid A Y as *a* is to *c*, and the solid A Y is to the solid A X as *c* is to *d*; *ex æquali* the solid A B is to the solid A X, or C D, which is equal to it, as *a* is to *d*. But the ratio of *a* to *d* is said to be compounded (V. *Def.* A) of the ratios of *a* to *b*, *b* to *c*, and *c* to *d*, which are the same with the ratios of the sides M A to A P, N A to A Q, and O A to A R, each to each: and the sides, A P, A Q and A R are equal to the sides D L, D K, and D H, each to each. Therefore the solid A B has to the solid C D the ratio which is the same with that which is compounded of the ratios of the sides A M to D L, A N to D K, and A O to D H. Q. E. D.



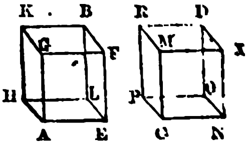
This proposition was introduced into this Book by Dr. Simson.

PROP. XXXIV. THEOREM.

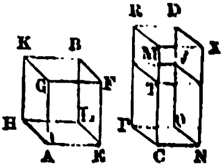
The bases and altitudes of two equal parallelopipeds, are reciprocally proportional: and conversely, if the bases and altitudes of two parallelopipeds be reciprocally proportional, they are equal.

Let A B and *C D* be two equal parallelopipeds. Their bases and *altitudes are reciprocally* proportional.

First, let their insisting straight lines A G, E F, L B, and H K; C M, N X, O D, and P R be at right angles to their bases. The base E H is to the base N P as C M is to A G.



If the base E H be equal to the base N P; because the solid A B is (*Hyp.*) equal to the solid C D, C M is equal to A G. For, if the bases E H and N P be equal, but the altitudes A G and C M not equal, neither is the solid A B equal to the solid C D. But the solids (*Hyp.*) are equal. Therefore the altitude C M is not unequal to the altitude A G; that is, C M is equal to A G. Wherefore the base E H is to the base N P, as C M is to A G. If the bases E H and N P be not equal, let E H be greater than the other. Because the solid A B is equal to the solid C D, C M is greater than A G. For, if not, the solids A B and C D are unequal. But (*Hyp.*) they are equal. Make C T equal to A G, and complete the parallelopiped C V, of which the base is N P, and altitude C T. Because the solid A B is equal to the solid C D, the solid A B is to the solid C V, as (V. 7) the solid C D is to the solid C V. But the solid A B is to the solid C V, as (XI. 32) the base E H is to the base N P; for the solids A B and C V are of the same altitude. Because the solid C D is to the solid C V, as (XI. 25) the base M P is to the base P T, and as (VI. 1) M C is to C T; and C T is equal to A G. Therefore the base E H is to the base N P, as M C is to A G. Wherefore the bases and altitudes of the parallelopipeds A B and C D are reciprocally proportional.



Conversely, let the bases and altitudes of the parallelopipeds A B and C D be reciprocally proportional. They are equal.

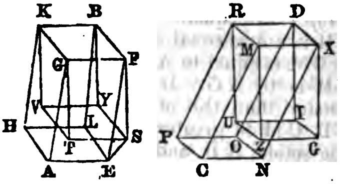
First, let their insisting straight lines be at right angles to their bases; and let the base E H be to the base N P, as C M is to A G. Let the base E H be equal to the base N P. Because E H is to N P, as the altitude of the solid C D is to the altitude of the solid A B. Therefore the altitude of C D is equal (V. A) to the altitude of A B. But parallelopipeds upon equal bases, and of the same altitude, are equal (XI. 31) to one another. Therefore the solid A B is equal to the solid C D. But let the bases E H and N P be unequal, and let E H be the greater. Because the base E H is to the base N P, as C M, the altitude of the solid C D, is to A G the altitude of A B, C M is greater (V. A) than A G. Make C T equal to A G, and complete the solid C V. Because the base E H is to the base N P, as C M is to A G, and A G is equal to C T. Therefore the base E H is to the base N P, as M C to C T. But the base E H is to the base N P, (XI. 32) as the solid A B is to the solid C V; for the solids A B and C V are of the same altitude: and M C is to C T, as (VI. 1) the base M P is to the base P T, and as the solid C D is to the solid (XI. 25) C V. Therefore the solid A B is to the solid C V, as the solid C D is to the solid C V; that is, each of the solids A B and C D has the same ratio to the solid C V. Therefore the solid A B is equal (V. 9) to the solid C D.

Secondly, let their insisting straight lines F E, B L, G A, and K H

X N, D O, M C, and R P be not at right angles to their bases. The base E H is to the base N P, as the altitude of C D is to the altitude of A B.

From the points F, B, K, and G; X. D, R, and M, draw perpendiculars to the planes in which are the bases E H and N P, meeting those planes in the points S, Y, V, and T; Q, I, U, and Z; and complete the solids F V and X U, which are parallelopipeds (XI. 31).

Because the solid A B is equal to the solid C D, and the solid A B is equal (XI. 29 or 30) to the solid B T, for they are upon the same base F K, and of the same altitude; and the solid C D is equal (XI. 29 or 30) to the solid D Z, being upon the same base X R, and of the same altitude. Therefore the solid B T is equal to the solid D Z. But the bases and altitudes of equal parallelopipeds of which the insisting straight lines are at right angles to their bases, are reciprocally proportional. Therefore the base F K is to the base X R, as the altitude of the solid D Z is to the altitude of the solid B T: and the base F K is equal to the base E H, and the base X R to the base N P. Therefore, the base E H is to the base N P, as the altitude of the solid D Z is to the altitude of the solid B T: but the altitudes of the solids D Z and D C, as also of the solids B T and B A, are the same. Therefore the base E H is to the base N P, as the altitude of the solid C D is to the altitude of the solid A B; that is, the bases and altitudes of the parallelopipeds A B and C D are reciprocally proportional.



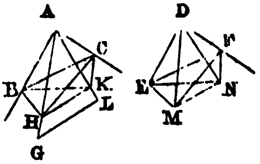
Conversely, let the bases of the solids A B and C D be reciprocally proportional to their altitudes, viz., the base E H to the base N P, as the altitude of the solid C D to the altitude of the solid A B. The solid A B shall be equal to the solid C D.

The same construction being made; because the base E H is to the base N P, as the altitude of the solid C D is to the altitude of the solid A B; and the base E H is equal to the base F K, and N P to X R. Therefore the base F K is to the base X R, as the altitude of the solid C D to the altitude of A B. But the altitudes of the solids A B and B T are the same, as also of C D and D Z. Therefore the base F K is to the base X R, as the altitude of the solid D Z is to the altitude of the solid B T. Wherefore the bases of the solids B T and D Z are reciprocally proportional to the altitudes: and their insisting straight lines are at right angles to their bases. Therefore, as was before proved, the solid B T is equal to the solid D Z. But B T is equal (XI. 29 or 30) to B A, and D Z to D C, because they are upon the same bases, and of the same altitude. Therefore the solid A B is equal to the solid C D. Therefore the bases, &c. Q. E. D.

PROP. XXXV. THEOREM.

If, from the vertices of two equal plane angles, there be drawn two straight lines elevated above the planes in which the angles are, and containing equal angles with the sides of those angles, each to each; and if in the lines above the planes there be taken any points, and from them perpendiculars be drawn to those planes; and if from the points in which they meet the planes, straight lines be drawn to the vertices of the two equal plane angles; these straight lines make equal angles with the straight lines above the planes.

Let B A C and E D F be two equal plane angles; and from the points ι and D let the straight lines A G and D M be drawn above the planes f the angles, making equal angles with their sides, viz., G A B equal to IDE, and G A C to M D F. From G and M, any points in the straight nes A G and D M, let perpendiculars G L and M N be drawn (XI. 11) to ie planes B A C and E D F meeting them in the points L and N; and t A L and M N be joined. The angle G A L is equal to the angle M D N. Make A H equal to D M, and through H draw H K parallel to G L in ie plane A G L, and meeting A L in K.



Because G L is perpendicular to the lane B A C, H K is perpendicular to the ime plane. Because the solid angle at A, contained by three plane angles B A C, A H, H A C, which are equal, each to ich, to the three plane angles E D F, D M, M D F, which contain the solid igle at D. Therefore the solid angles at A and D are equal, and incide with one another (XI. B); that is, if the plane angle B A C be pplied to the plane angle E D F, the straight line A H coincides with M. Because A H is equal to D M, the point H coincides with the oint M. Therefore H K, which is perpendicular to the plane B A C, incides with M N (XI. 13) which is perpendicular to the plane E D F, id H K is equal to M N. Because the points A and K coincide with ie points D and N, the straight line A K coincides with the straight ne D N. Therefore the triangle A H K coincides with the triangle M N, and the angle H A K with the angle M D N. Wherefore the igle G A L is equal to the angle M D N. Therefore, if from the vertices, c. Q. E. D.

Corollary.—From this it is manifest, that if from the vertices of two ual plane angles, there be drawn two equal straight lines containing ual angles with the sides of the angles each to each; the perpendicurs drawn from the extremities of the equal straight lines to the planes the first angles are equal to one another.

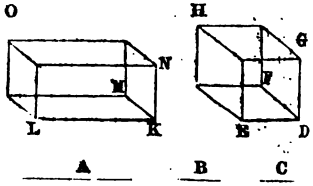
PROP. XXXVI. THEOREM.

three straight lines be proportionals, the parallelopiped described from all three, as its sides, is equal to the equilateral parallelopiped described from the mean proportional, one of the solid angles of which is contained by three plane angles equal, each to each, to the three plane angles containing one of the solid angles of the other.

Let A, B, C be three proportionals, viz., A to B, as B to C. The parallelopiped described from A, B, C as its edges shall be equal to the equilateral parallelopiped described from B, as its common edge, equiangular to the other.

Take a solid angle D contained by three plane angles E D F, F D G, and G D E; and make each of the straight lines E D, D F, and D G equal to B, and complete the parallelopiped D H. Take any straight line L K equal to A, and at the point K in the straight line L K, make (XI. 26) a solid angle contained by the three plane angles L K M, M K N and N K L, equal to the angles E D F, F D G, and G D E, each to each; make K N equal to B, and K M equal to C; and complete the parallelopiped K O.

Because, A is to B, as B is to C, and A is equal to L K, and B to each of the straight lines D E, D F, and C to K M. Therefore L K is to E D, as D E is to K M; that is, the sides about the equal angles are reciprocally proportional; therefore the parallelogram L M is equal (VI. 14) to the parallelogram E F. Because E D F and L K M are two equal plane angles, and the two equal straight lines D G and K N are drawn from their vertices above their planes; and contain equal angles with their sides. Therefore the perpendiculars from the points G and N to the planes E D F and L K M are equal to (XI. 35 *Cor.*) one another. Wherefore the solids K O and D H are of the same altitude: and they are upon equal bases L M, and E F. Therefore the parallelopiped K O is equal (XI. 31) to the parallelopiped D H; and the solid K O is described from the three straight lines A, B, and C, as its edges, and the solid D H from the straight line B as its common edge. Therefore if three straight lines, &c. Q. E. D.



Exercise.—Demonstrate the converse of this proposition.

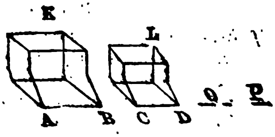
PROP. XXXVII. THEOREM.

If four straight lines be proportionals, the similar parallelopipeds similarly described from them as edges are also proportionals; and, conversely, if the similar parallelopipeds similarly described from four straight lines as edges be proportionals, the straight lines are proportionals.

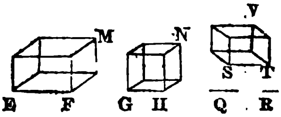
Let the four straight lines A B, C D, E F and G H, be proportionals, viz., A B to C D, as E F to G H; and let the similar parallelopipeds A K, C L, E M and G N be similarly described from them as edges. A K is to C L, as E M is to G N.

Make (VI. 11) A B, C D, O and P continual proportionals, as also E F, G H, Q and R.

Because A B is to C D, as E F to G H; and C D is (V. 11) to O, as G H is to Q; and O is to P, as Q is to R. Therefore, *ex æquali* (V. 22) A B is to P, *as E F is to R.* But A B is to P, as (XI. *Cor.* 33) *the* solid A K, is to the solid *C L; and E F* is to R, as (XI. Cor. 33) *the solid E M is* to the solid G N. There-



fore (V. 11) the solid A K is to the solid C L, as the solid E M is to the solid G N.



Next, let the solid A K be to the solid C L, as the solid E M to the solid G N. The straight lines A B, C D, E F, and G H, are proportionals.

Take A B to C D, so E F to S T, and from S T as edge, describe (XI. 27) a parallelopiped S V similar and similarly situated to either of the parallelopipeds, E M or G N.

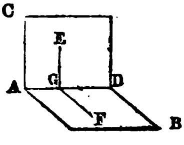
Because A B is to C D, as E F is to S T, and that from A B and C D as edges the parallelopipeds A K and C L are similarly described; and the parallelopipeds E M and S V from the straight lines E F and S T as edges. Therefore A K is to C L, as E M is to S V. But (*Hyp.*) A K is to C L, as E M is to G N. Therefore G N is equal (V. 9) to S V. But it is similar and similarly situated to S V. Therefore the planes which contain the solids G N and S V are similar and equal, and their homologous sides G H and S T equal to one another. Because A B is to C D, as E F is to S T, and S T is equal to G H. Therefore A B is to C D, as E F is to G H. Therefore, if four straight lines, &c. Q. E. D.

PROP. XXXVIII. THEOREM.

If one plane be perpendicular to another, a straight line drawn from a point in the one perpendicular to the other, meets their common section.

Let the plane C D be perpendicular to the plane A B, and let A D be their common section; if any point E be taken in the plane C D, the perpendicular, drawn from E to the plane A B, meets A D.

From E draw E G in the plane C D perpendicular to A D (I. 12) and in the plane A B, from the point G, draw G F perpendicular to A D (I. 11).



Because E G is perpendicular to G F (XI. *Def.* 3), and also to A D (*Const.*) Therefore E G is perpendicular to the plane A B (XI. 4). But from the point E, no other straight line can be drawn perpendicular to A B, than E G (XI. 13). Therefore the perpendicular drawn from E to the plane A B meets A D. Wherefore, if one plane be perpendicular, &c. Q. E. D.

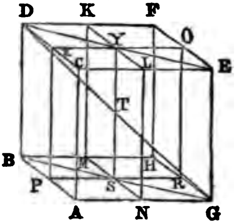
Dr. Simson considers this proposition as an interpolation in this Book, and quite out of place.

PROP. XXXIX. THEOREM.

In a parallelopiped, if the sides of two of the opposite planes be each bisected, the common section of the planes passing through the points of bisection and any diagonal of the parallelopiped, bisect each other.

Let the sides of the opposite planes C F and A H, of the parallelopiped A F, be bisected in the points K, L, M, and N; and X, O, P, and R. Join K L, M N, X O, and P R. Because D K and C L are equal and parallel, K L is parallel (I. 33) to D C. For the same reason, M N is parallel to B A. But B A is parallel to D C. Because K L and B A are each of them parallel to D C, and not in the same plane with it, K L is parallel (XI. 9) to B A; and because K L and M N are each of them parallel to

B A, and not in the same plane with it, K L is parallel (XI. 9) to M N. Therefore K L and M N are in one plane. In like manner it may be proved, that X O and P R are in one plane. Let Y S be the common section of the planes K N and X R; and D G the diagonal of the parallelopiped A F. The straight lines Y S and D G meet and bisect each other. Join D Y, Y E, B S, and S G.



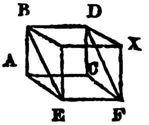
Because D X is parallel to O E, the alternate angles D X Y and Y O E are equal (I. 29). Because D X is equal to O E and X Y to Y O, and they contain equal angles, the base D Y is equal (I. 4) to the base Y E, and the other angles are equal. Therefore the angle X Y D is equal to the angle O Y E, and D Y E is a straight (I. 14) line. For the same reason B S G is a straight line, and B S is equal to S G. Because C A is equal and parallel to D B, and also to E G. Therefore D B is equal and parallel (XI. 9) to E G; and D E and B G join their extremities. Therefore D E is equal and parallel (I. 33) to B G. But D G and Y S drawn joining them are in one plane. Therefore D G and Y S must meet one another; let them meet in T. Because D E is parallel to B G, the alternate angles E D T and B G T are (XI. 29) equal. But the angle D T Y is equal (I. 15) to the angle G T S. Wherefore in the triangles D T Y and G T S there are two angles in the one equal to two angles in the other, and one side equal to one side, opposite to equal angles in each, viz., D Y to G S; for they are the halves of D E and B G. Therefore the remaining sides are equal (I. 26), each to each. Wherefore D T is equal to T G, and Y T to T S. Therefore, if in a parallelopiped, &c. Q. E. D.

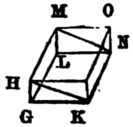
PROP. XL. THEOREM.

If two triangular prisms of the same altitude, have the base of one a parallelogram, and the base of the other a triangle; and if the parallelogram be double of the triangle, the prisms are equal.

Let the prisms A B C D E F and G H K L M N be of the same altitude, of which the first is contained by the two triangles A B E and C D F, and the three parallelograms A D, D E, and E C; and the other by the two triangles G H K and L M N, and the three parallelograms L H, H N, and N G; and let one of them have a parallelogram A F, and the other a triangle G H K, for its base: if the parallelogram A F be double of the triangle G H K, the prism A B C D E F is equal to the prism G H K L M N.

Complete the solid parallelopipeds A X and G O.





Because the parallelogram A F is double of the triangle G H K; and the parallelogram H K double (I. 34) of the same triangle. Therefore the parallelogram A F is equal to H K. But parallelopipeds upon equal bases, and of the same altitude, are equal (XI. 31) to one another. Therefore the solid A X is equal to the solid G O. But the prism A B C D E F is half (XI. 28) of the solid A X; and the prism G H K L M N half (XI. 28) of *the solid G O.* Therefore the prism A B C D E F is equal to the prism *G H K L M N.* Wherefore, if two triangular prisms, &c. Q. E. D.

BOOK XII.

DEFINITIONS.

I.

A PYRAMID is said to be inscribed in a cone, when its base is inscribed in the base of the cone, and both solids have a common vertex; and, a cone is said to be described about a pyramid, when its base is described about the base of the pyramid, and both solids have a common vertex.

II.

A cone is said to be inscribed in a pyramid, when its base is inscribed in the base of the pyramid, and both solids have a common vertex; and, a pyramid is said to be described about a cone, when its base is described about the base of the cone, and both solids have a common vertex.

III.

A prism is said to be inscribed in a cylinder when its bases are inscribed in the bases of the cylinder; and, a cylinder is said to be described about a prism, when its bases are described about the bases of the prism.

IV.

A cylinder is said to be inscribed in a prism when its bases are inscribed in the bases of the prism; and, a prism is said to be described about a cylinder, when its bases are described about the bases of the cylinder.

V.

A polyhedron is said to be inscribed in a sphere when the vertices of its solid angles are in the superficies of the sphere; and, a sphere is said to be described about a polyhedron when the superficies of the sphere passes through the vertices of its solid angles.

VI.

A sphere is said to be inscribed in a polyhedron when its superficies touches the faces or planes of the polyhedron; and, a polyhedron is said to be described about a sphere, when its faces or planes touch the superficies of the sphere.

POSTULATES.

I.

Let it be granted that a square or other rectilineal figure may contain the same area or space as a circle; or, that to two squares and a circle there may be a fourth proportional.

II.

Let it be granted that to two triangles and a pyramid erected on one of them, or to two similar rectilineal figures and a solid erected on one of them, there may be a fourth proportional; and that to any three solids there may be a fourth proportional.

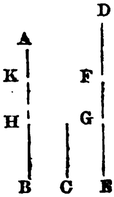
The preceding definitions and postulates are not laid down in Euclid's Elements. But as they are understood and taken for granted in this Book, they are formally inserted here for the benefit of the learner.

The following Lemma is the first proposition of the Tenth Book of Euclid's Elements, and is usually inserted here, as being necessary to the understanding of the demonstration of some of the propositions of this Book.

LEMMA I.

If from the greater of two unequal magnitudes of the same kind, there be taken more than its half, and from the remainder more than its half; and so on: there will at length remain a magnitude less than the least of the proposed magnitudes.

Let A B and C be two unequal magnitudes, of the same kind, of which A B is the greater. If from A B there be taken more than its half, and from the remainder more than its half, and so on; there will at length remain a magnitude less than C.



For C may be multiplied so as at length to become greater than A B. Let it be so multiplied, and let D E its multiple be greater than A B; also let D E be divided into parts D F, F G, and G E, each equal to C. From A B take B H greater than its half, and from the remainder A H take H K greater than its half, and so on, until there be as many divisions in A B as there are in D E: and let the divisions in A B be A K, K H, and H B; and the divisions in D E be D F, F G, and G E.

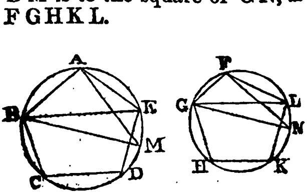
Because D E is greater than A B, and E G taken from D E is not greater than its half, but B H taken from A B is greater than its half. Therefore the remainder G D is greater than the remainder H A. Again, because G D is greater than H A, and that G F is not greater than the half of G D, but H K is greater than the half of H A. Therefore the remainder F D is greater than the remainder A K. But F D is equal to C. Therefore C is greater than A K; that is, A K is less than C. Wherefore, if from the greater, &c. Q. E. D.

Corollary.—If only the halves be taken away, the same thing may in the same way be demonstrated.

PROP. I. THEOREM.

Similar polygons inscribed in circles, are to one another as the squares of their diameters.

Let A B C D E and F G H K L be two circles having the similar polygons A B C D E and F G H K L inscribed in them; and let B M and G N be the diameters of the circles. The square of B M is to the square of G N, as the polygon A B C D E is to the polygon F G H K L.



Join B E, A M, G L, and F N. Because the polygon A B C D E is similar to the polygon F G H K L, the angle B A E is equal to the angle G F L, and B A is to A E as G F is to F L. Therefore the two triangles B A E and G F L (VI. 6) are equiangular; and the angle A E B is equal to the angle F L G. But the angle A E B is equal (III. 21) to the angle *A M B, because they* stand upon the same arc. For the same reason, the angle F L G is equal to the angle F N G. Therefore also the angle A M B is equal to the angle F N G; and the angle B A M is equal to the (III. 31) angle G F N. Therefore the remaining angles in the triangles A B M and F G N are equal, and they are equiangular to one another. Wherefore B M is to G N, as (VI. 4) B A is to G F. And the duplicate ratio of B M to G N, is the same (V. 5 and 22, *Def.* 10) with the duplicate ratio of B A to G F. But the ratio of the square of B M to the square of G N, is the duplicate (VI. 20) ratio of that which B M has to G N; and the ratio of the polygon A B C D E to the polygon F G H K L is the duplicate (VI. 20) of that which B A has to G F. Therefore the square of B M is to the square of G N, as the polygon A B C D E is to the polygon F G H K L. Wherefore, similar polygons, &c. Q. E. D.

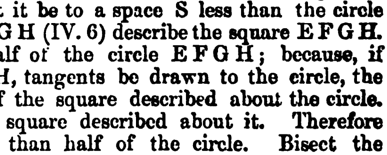
Corollary—Similar polygons are to one another as the squares of the radii of their inscribed or circumscribed circles, or as the squares of the chords of similar segments.

PROP. II. THEOREM.

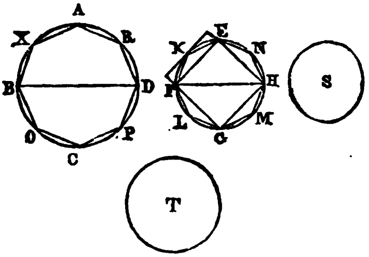
Circles are to one another as the squares of their diameters.

Let A B C D and E F G H be two circles, and B D and F H their diameters. The square of B D is to the square of F H as the circle A B C D is to the circle E F G H.

For, if not, the square of B D is to the square of F H, as the circle A B C D is to some space either less than the circle E F G H, or greater than it (XII. *Post.* 1). First, let it be to a space S less than the circle E F G H; and in the circle E F G H (IV. 6) describe the square E F G H. This square is greater than half of the circle E F G H; because, if through the points E, F, G, and H, tangents be drawn to the circle, the square E F G H is half (I. 41) of the square described about the circle. But the circle is less than the square described about it. Therefore the square E F G H is greater than half of the circle. Bisect the arcs E F, F G, GH, and H E, at the points K, L, M, and N, and join E K, K F, F L, L G, G M, M H, H N, and N E. Each of the triangles E K F and F L G is greater than half of the segment which contains it. For if straight lines touching the circle be drawn through the points K, L, M, and N, and the parallelograms upon the straight lines E F, F G, G H, and H E be completed, each of the triangles E K F, F L G, G M H, and H N E is the half (I. 41) of the parallelogram which contains it. But every segment is less than the parallelogram which contains it. Therefore each of the triangles E K F, F L G, G M H, and H N E is greater than half the segment which contains it. Again, if the arcs E K, K F, &c. be bisected, and their extremities be joined; and so on: there will at length remain segments of the circle, which taken together are less than the excess of the circle E F G H above the space S. For (XII. Lemma 1) if from the greater of two unequal magnitudes there be taken more than its half, and from the remainder more than its half, and so on, there will at length remain a magnitude less than the least of the proposed magnitudes. Let the segments E K, K F, F L, L G, G M, M H, H N, and N E be those which remain, and are together less than the excess of the circle



E F G H above S. Therefore the rest of the circle, viz., the polygon E K F L G M H N is greater than the space S. Describe likewise in the circle A B C D the polygon A X B O C P D R similar to the polygon E K F L G M H N. Because the square of B D is to the square of F H (XII. 1) as the polygon A X B O C P D R is to the polygon E K F L G M H N. But the square of B D is to the square of F H (*Hyp.*) as the circle A B C D is to the space S. Therefore the circle A B C D is to the space S, as (V. 11) the polygon A X B O C P D R is to the polyglon E K F L G M H N. But the circle A B C D is greater than the polygon contained in it. Therefore the space S is greater (V. 14) than the polygon E K F L G M H N; but it is also less, as has been proved, which is impossible. Therefore the square of B D is not to the square of F H, as the circle A B C D is to any space less than the circle E F G H. In the same manner, it may be demonstrated, that the square of F H is not to the square of B D, as the circle E F G H is to any space less than the circle A B C D. Neither is the square of B D to the square of F H, as the circle A B C D to any space greater than the circle E F G H. For, if possible, let the square of B D be to the square of F H, as the circle A B C D is to T, a space greater than the circle E F G H. Therefore, inversely, the square of F H is to the square of B D, as the space T is to the circle A B C D. But the space T is to the circle A B C D, as the circle E F G H is to some space, which must be less (V. 14) than the circle A B C D, because the space T is greater. (*Hyp.*) than the circle E F G H. Therefore the square of F H is to the square of B D, as the circle E F G H is to a space less than the circle A B C D, which has been proved to be impossible. Therefore the square of B D is not to the square of F H as the circle A B C D is to any space greater than the circle E F G H. But it has been proved, that the square of B D is not to the square of F H, as the circle A B C D is to any space less than the circle E F G H. Therefore, the square of B D is to the square of F H, as the circle A B C D is to the circle E F G H. Therefore, circles are, &c. Q. E. D.



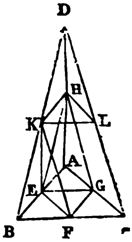
Corollary 1.—Similar polygons are to one another as their inscribed or circumscribed circles.

Corollary 2.—Circles are to one another, as the squares of their radii, or as the squares of the chords of similar segments; and so are these segments.

PROP. III. THEOREM.

Every triangular pyramid may be divided into two equal and similar triangular pyramids similar to the whole pyramid, and two equal prisms which are together greater than half of the whole pyramid.

Let *A B CD* be a pyramid of which the base is the triangle A B C and its vertex the point D. The pyramid may be divided into two equal and similar triangular pyramids similar to the whole pyramid, and two equal prisms which are together greater than half of the whole pyramid.



Bisect A B, B C, C A, A D, D B, and D C, at the points E, F, G, H, K and L; and join E H, E G, G H, H K, K L, L H, E K, K F, F G, and E F.

Because A E is equal to E B, and A H to H D, H E is parallel (VI. 2) to D B. For the same reason, H K is parallel to A B. Therefore H E B K is a parallelogram, and H K is equal (I. 34) to E B. But E B is equal to A E. Therefore also A E is equal to H K, and A H is equal to H D. Wherefore E A and A H are equal to K H and H D, each to each; and the angle E A H is equal (I. 29) to the angle K H D. Therefore the base E H is equal to the base K D, and the triangle A E H is equal (I. 4) and similar to the triangle H K D. For the same reason, the triangle A G H is equal and similar to the triangle H L D. Again, because E H and H G, which meet one another, are parallel to K D and D L, that meet one another, but are not in the same plane with them, they contain equal (XI. 10) angles. Therefore the angle E H G is equal to the angle K D L. Because E H and H G are equal to K D and D L, each to each, and the angle E H G equal to the angle K D L. Therefore the base E G is equal to the base K L, and the triangle E H G equal (I. 4) and similar to the triangle K D L. For the same reason, the triangle A E G is also equal and similar to the triangle H K L. Therefore the pyramid A E G H is equal (XI. C) and similar to the pyramid K H L D. Because H K is parallel to A B, a side of the triangle A D B, the triangle A D B is equiangular to the triangle H D K, and their sides are (VI. 4) proportionals. Therefore the triangle A D B is similar to the triangle H D K. For the same reason, the triangle D B C is similar to the triangle D K L; the triangle A D C to the triangle H D L; and the triangle A B C to the triangle A E G. But the triangle A E G was proved to be similar to the triangle H K L. Therefore the triangle A B C is similar (VI. 21) to the triangle H K L. Wherefore the pyramid A B C D is similar (XI. B, and XI. *Def.* 11) to the pyramid H K L D. But the pyramid H K L D is similar, as has been proved, to the pyramid A E G H. Therefore the pyramid A B C D is similar to the pyramid A E G H. Wherefore each of the pyramids A E G H and H K L D is similar to the whole pyramid A B C D. Because B F is equal to F C, the parallelogram E B F G is double (I. 41) of the triangle G F C. But the prism E B F G H K having the parallelogram E B F G for its base, is equal (XI. 40) to the prism G F C H K L, having the triangle G F C for its base; for they are of the same altitude, viz., the distance between the parallel (XI. 15) planes A B C and H K L. But each of these prisms is greater than either of the pyramids A E G H and H K L D. For the prism E B F G H K on the base E B F G, is greater than the pyramid E B F K, on the base E B F. But the pyramid E B F K is equal (XI. C) to the pyramid A E G H; because they are contained by equal and similar planes. Therefore the prism E B F G H K *is greater than the pyramid* A E G H. But the prism E B F G H K is equal to the prism GFCHKL; and the pyramid AEGH is equal to the pyramid HKLD. Therefore the two prisms EBFGH and GFCHKL are greater than the two pyramids AEGH and HKLD. Wherefore the whole pyramid ABCD is divided into two equal triangular pyramids similar to one another and to the whole pyramid; and into two equal prisms which are together greater than half of the whole pyramid. Q. E. D.

Corollary.—A section of a triangular pyramid parallel to its base is similar to the base, and conversely.

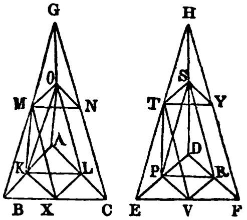
PROP. IV. THEOREM.

If two triangular pyramids of the same altitude be each divided into two equal pyramids similar to the whole pyramid, and also into two equal prisms; and if each of these pyramids be divided in the same manner as the first two, and so on simultaneously; the base of one of the first two pyramids is to the base of the other, as all the prisms in the one is to all the prisms in the other.

Let the two triangular pyramids be of the same altitude ABCG and DEFH, and be each divided into two equal pyramids similar to the whole, and into two equal prisms; and let each of the pyramids thus made be supposed to be divided in the same manner, and so on simultaneously. The base ABC is to the base DEF, as all the prisms in the pyramid ABCG is to all the prisms in the pyramid DEFH.

Make the same construction as in the foregoing proposition.

Because BX is equal to XC, and AL to LC. Therefore XL is parallel (VI. 2) to AB, and the triangle ABC is similar to the triangle LXC. For the same reason, the triangle DEF is similar to RVF. Because BC is double of CX, and EF double of FV. Therefore (V. C) BC is to CX, as EF to FV. But upon BC and CX are described the similar and similarly situated rectilineal figures ABC and LXC; and upon EF and FV, the similar figures DEF and RVF. Therefore, the triangle ABC is to the triangle LXC (VI. 22), as the triangle DEF is to the triangle RVF, and, by permutation, the triangle ABC is to the triangle DEF, as the triangle LXC is to the triangle RVF. Again, because the planes ABC and OMN, as also the planes DEF and STY, are parallel (XI. 15), and GC and HF are bisected at the points N and Y (*Const.*) the equal (*Hyp.*) perpendiculars drawn from the points G and H to the bases ABC and DEF will be bisected (XI. 17) by the planes OMN and STY. Therefore the prisms LXCOMN, RVFSTY are of the same altitude. Wherefore the base LXC is to the base RVF, that is, the triangle ABC is to the triangle DEF, as the prism LXCOMN is to the prism RVFSTY. Because the two prisms in the pyramid ABCG are equal to one another, and the two prisms in the pyramid DEFH are also equal to one another. Therefore the prism KBXLMO is to the prism LXCOMN, as the prism PEVRTS is to the prism RVFSTY. But, *componendo*, the prisms KBXLMO and LXCOMN *together, are to* the prism LXCOMN, as the prisms PEVRTS *and RVFSTY*, together, are to the prism RVFSTY. Therefore, *permutando,* the prisms KBXLMO and LXCOMN are to the *prisms PEVRTS* and RVFSTY, as the prism LXCOMN is to the prism R V F S T Y. But it has been proved that the prism L X C O M N is to the prism R V F S T Y, as the base A B C is to the base D E F. Therefore, the base A B C is to the base D E F, as the two prisms in the pyramid A B C G is to the two prisms in the pyramid D E F H. In like manner, if the pyramids now made, for example, the two O M N G, S T Y H, be similarly divided. The base O M N is to the base S T Y, as the two prisms in the pyramid O M N G is to the two prisms in the pyramid S T Y H. But the base O M N is to the base S T Y, as the base A B C is to the base D E F. Therefore, the base A B C is to the base D E F, as the two prisms in the pyramid A B C G is to the two prisms in the pyramid D E F H; and as the two prisms in the pyramid O M N G are to the two prisms in the pyramid S T Y H; and so are all four to all four; and the same may be shown of the prisms made by dividing the pyramids A K L O and D P R S, and of all made by simultaneous division. Wherefore, if two triangular pyramids, &c. Q. E. D.



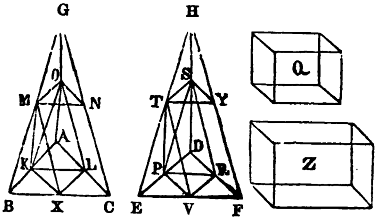
PROP. V. THEOREM.

Triangular pyramids of the same altitude, are to one another as their bases.

Let the pyramids A B C G and D E F H be of the same altitudes. The base A B C is to the base D E F, as the pyramid A B C G is to the pyramid D E F H.

For, if not, the base A B C is to the base D E F, as the pyramid A B C G is to a solid either less than the pyramid D E F H, or greater than it (XI. *Post* 2). First, let it be to a solid less than it, viz. to the solid Q. Divide the pyramid D E F H into two equal pyramids, similar to the whole, and into two equal prisms. Therefore these two prisms are greater (XII. 3) than the half of the whole pyramid. Again, let the pyramids made by this division be similarly divided, and so on (XII. *Lemma* 1) until the pyramids which remain undivided in the pyramid D E F H be, all of them together, less than the excess of the pyramid D E F H above the solid Q. Let these, for example, be the pyramids D P R S and S T Y H. Therefore the prisms, which make the rest of the pyramid D E F H, are greater than the solid Q. Divide, likewise, the pyramid A B C G in the same manner, and into as many parts, as the pyramid D E F H. Therefore, the base A B C is to the base D E F (XII. 4), as the prisms in the pyramid A B C G are to the prisms in the pyramid D E F H. But the base A B C is to the base D E F, as (*Hyp.*) the pyramid A B C G is to the solid Q. Therefore, the pyramid A B C G is to the solid Q, as the prisms in the pyramid A B C G are to the prisms in the pyramid D E F H. But the pyramid A B C G is greater than the prisms contained in it. Wherefore (V. 14) also the solid Q is greater than the prisms in the pyramid D E F H; but it is also less, which is impossible. Therefore the base A B C is not to the base D E F, as the pyramid A B C G to any solid less than the pyramid D E F H. In the same manner it may be demonstrated, that the base D E F is not to the base A B C, as the pyramid D E F H to any solid less than the pyramid

A B C G. Neither can the base A B C be to the base D E F, as the pyramid A B C G to any solid greater than the pyramid D E F H. For, if it be possible, let this greater solid be Z. Because the base A B C is to the base D E F as the pyramid A B C G is to the solid Z. Therefore, by inversion, the base D E F is to the base A B C, as the solid Z is to the pyramid A B C G. But the solid Z is to the pyramid A B C G, as the pyramid D E F H to some solid (XII. *Post.* 2) less (V. 14) than the pyramid A B C G, because the solid Z is greater than the pyramid D E F H. Therefore, the base D E F is to the base A B C, as the pyramid D E F H is to a solid less than the pyramid A B C G; the contrary to which has been proved. Therefore the base A B C is not to the base D E F, as the pyramid A B C G is to any solid greater than the pyramid D E F H. And it has been proved that neither is the base A B C to the base D E F, as the pyramid A B C G is to any solid less than the pyramid D E F H. Therefore, the base A B C is to the base D E F, as the pyramid A B C G is to the pyramid D E F H. Wherefore triangular pyramids, &c. Q. E. D.



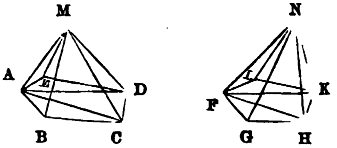
PROP. VI. THEOREM.

Polygonal pyramids of the same altitude, are to one another as their bases.

Let the pyramids A B C D E M and F G H K L N be of the same altitude. The base A B C D E is to the base F G H K L, as the pyramid A B C D E M is to the pyramid F G H K L N.

Divide the base A B C D E into the triangles A B C, A C D, and A D E, and the base F G H K L into the triangles F G H, F H K, and F K L; and let planes pass through A D M, A C M, F K N, and F H N. Therefore the pyramid A B C D E M contains as many triangular pyramids having the common vertex M, as the pyramid F G H K L N contains, having the common vertex N.

Because the triangle A B C is to the triangle F G H, as (XII. 5) the pyramid A B C M is to the pyramid F G H N; the triangle A C D is to the triangle F G H, as the pyramid A C D M is to the pyramid F G H N; and the triangle A D E is to the triangle F G H, as the pyramid A D E M is to the pyramid F G H N. Therefore the base A B C D E is to the base F G H, as the pyramid A B C D E M is to the pyramid F Y G H N (V. 24, *Cor.* 2); and the base F G H K L is to the base F G H, as the pyramid F G H K L N is to the pyramid F G H N. But, by inversion, the base F G H is to the base F G H K L, as the pyramid F G H N is to the pyramid F G H K L N. Wherefore the base A B C D E is to the base F G H, as the pyramid A B C D E M to the pyramid F G H N; and the base



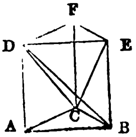
FGH is to the base F G H K L, as the pyramid F G H N is to the pyramid F G H K L N. Therefore, *ex æquali* (V. 22), the base A B C D E is to the base F G H K L, as the pyramid A B C D E M is to the pyramid F G H K L N. Therefore polygonal pyramids, &c. Q. E. D.

PROP. VII. THEOREM.

Every triangular prism may be divided into three equal triangular pyramids.

Let the prism A B C D E F be triangular. It may be divided into three equal triangular pyramids.

Join B D, E C, and C D. Because A B E D is a parallelogram, and B D its diagonal, the triangle A B D is equal (I. 34) to the triangle E B D. Therefore the pyramid A B D C is equal (XII. 5) to the pyramid E B D C. Because F B is a parallelogram and C E its diagonal, the triangle E C F is equal (I. 34) to the triangle E C B. Therefore the pyramid E C B D is equal to the pyramid E C F D. But the pyramid E C B D has been proved equal to the pyramid A B D C. Therefore the prism A B C D E F is divided into three equal pyramids having triangular bases, viz. the pyramids A B D C, E B D C, and E C F D. Wherefore every triangular prism, &c. Q. E. D.



COROLLARY 1.—From this it is manifest, that every pyramid is the third part of a prism having the same base and altitude. For if the base of the prism be any other figure than a triangle, it may be divided into prisms having triangular bases.

COROLLARY 2.—Prisms of equal altitudes are to one another as their bases; because the pyramids upon the same bases, and of the same altitude, are (XII. 6) to one another as their bases.

This proposition is the foundation of the rule for finding the solid content of a pyramid.

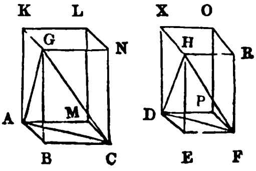
PROP. VIII. THEOREM.

Similar triangular pyramids, are to one another, in the triplicate ratio of their homologous sides.

Let the pyramids A B C G and D E F H be similar and similarly situated. The pyramid A B C G has to the pyramid D E F H, the triplicate ratio of that which the side B C has to the homologous side E F.

Complete the parallelograms A B C M, G B C N, and A B G K, and the parallelopiped B L. Complete also the parallelopiped E O.

Because the pyramid A B C G is similar to the pyramid D E F H, the angle A B C is equal (XI. *Def.* 11) to the angle D E F, the angle G B C to the angle H E F, and the angle A B G to the angle D E H; and A B is (VI. *Def.* 1) to B C, as D E is to E F; that is, the sides about the equal angles are proportionals. Therefore the parallelogram B M is similar to the parallelogram E P: for the same reason, the parallelograms B N and E R are similar to the parallelograms B K and E X. Therefore the three parallelograms B M, B N, and B K are similar to the three E P, E R, and E X. But the three



BM, BN, and BK, are equal and similar (XI. 24) to the three which are opposite to them; and the three EP, ER, and EX are equal and similar to the three opposite to them. Therefore the solids BL and EO are contained by the same number of similar planes; and their solid angles are (XI. B) equal. Therefore the solid BL is similar (XI. *Def.* 11) to the solid EO. But similar parallelopipeds have the triplicate (XI. 33) ratio of that which their homologous sides or edges have. Therefore the solid BL has to the solid EO the triplicate ratio of that which the side BC has to the homologous side EF. But the solid BL is to the solid EO as (V. 15), the pyramid ABCG is to the pyramid DEFH; for the pyramids are each the sixth part of the solids, since the prisms which are halves (XI. 28) of the parallelopipeds are triple (XII. 7) of the pyramids. Therefore, likewise the pyramid ABCG has to the pyramid DEFH, the triplicate ratio of that which BC has to the homologous side of EF. Wherefore similar triangular pyramids, &c. Q. E. D.

COROLLARY.—From this it is evident, that similar polygonal pyramids are to one another in the triplicate ratio of their homologous sides. For they may be divided into similar pyramids having triangular bases; because the similar polygons, which are their bases, may be divided into the same number of similar triangles homologous to the whole polygons. Therefore, as one of the triangular pyramids in the first polygonal pyramid is to one of the triangular pyramids in the other (V. 12), so are all the triangular pyramids in the one to all the triangular pyramids in the other; that is, as the one polygonal pyramid is to the other. But one triangular pyramid is to its similar triangular pyramid, in the triplicate ratio of their homologous sides. Therefore, the one polygonal pyramid has to the other, the triplicate ratio of that which one of the sides of the one has to the homologous side of the other.

This proposition is the foundation of the rule for the mensuration of similar solids and all regular polyhedrons.

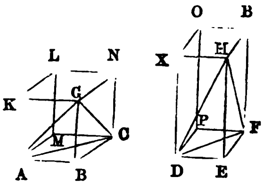
PROP. IX. THEOREM.

The bases and altitudes of equal triangular pyramids are reciprocally proportional: and triangular pyramids, of which the bases and altitudes are reciprocally proportional, are equal to one another.

First, let the pyramids ABCG and DEFH, be equal to one another. The base ABC is to the base DEF, as the altitude of the pyramid DEFH is to the altitude of the pyramid ABCG.

Complete the parallelograms AC, AG, GC, DF, DH, and HF; and the parallelopipeds BL and EO.

Because the pyramid ABCG is equal to the pyramid DEFH, and the solid BL is sextuple (XI. 28 and XII. 7) of the pyramid ABCG and the solid EO sextuple of the pyramid DEFH. Therefore the solid BL is equal (V. *Ax.* 1) to the solid EO. But the bases and altitudes of equal parallelopipeds are reciprocally proportional (XI. 34). Therefore as the base BM to the base EP, so is the altitude of the solid EO to the altitude of the solid BL. But the base BM is to the base EP, as (V. 15) the triangle ABC is to the triangle DEF. Therefore the triangle ABC is to the triangle DEF, as the altitude of the solid E O is to the altitude of the solid B L. But the altitude of the solid E H P O is the same with the altitude of the pyramid D E F H; and the altitude of the solid B G M L is the same with the altitude of the pyramid A B C G. Therefore the base A B C is to the base D E F, as the altitude of the pyramid D E F H is to the altitude of the pyramid A B C G. Wherefore, the bases and altitudes of the pyramids A B C G and D E F H are reciprocally proportional.



Next, let the bases and altitudes of the pyramids A B C G and D E F H be reciprocally proportional. The pyramid A B C G is equal to the pyramid D E F H.

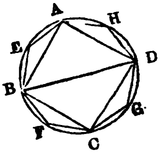
The same construction being made, because the base A B C is to the base D E F, as the altitude of the pyramid D E F H is to the altitude of the pyramid A B C G; and the base A B C is to the base D E F, as the parallelogram B M is to the parallelogram E P. Therefore the parallelogram B M is to the parallelogram E P, as the altitude of the pyramid D E F H is to the altitude of the pyramid A B C G. But the altitude of the pyramid D E F H is the same with the altitude of the parallelopiped E O; and the altitude of the pyramid A B C G is the same with the altitude of the parallelopiped B L. Therefore the base B M is to the base E P, as the altitude of the parallelopiped E O is to the altitude of the parallelopiped B L. Therefore the parallelopiped B L is equal to the parallelopiped E O (XI. 34). But the pyramid A B C G is the sixth part of the solid B L, and the pyramid D E F H is the sixth part of the solid E O. Therefore the pyramid A B C G is equal (V. *Ax.* 2) to the pyramid D E F H. Therefore the bases, &c. Q. E. D.

PROP. X. THEOREM

A cone is the third part of a cylinder, having the same base and altitude.

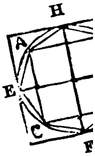
Let a cone and a cylinder have the same base, viz., the circle A B C D, and the same altitude. The cone is the third part of the cylinder, that is, the cylinder is triple of the cone.

If the cylinder be not triple of the cone, it must either be greater than the triple, or less than the triple. First, let it be greater than the triple. Inscribe the square A B C D in the circle: this square is greater (XII. 2) than the half of the circle A B C D. Upon the square A B C D erect a prism of the same altitude with the cylinder; this prism is greater than half of the cylinder. For, let a square be described about the circle, and let a prism be erected upon the square, of the same altitude as the cylinder. Because the inscribed square is half of that circumscribed; and upon these square bases are erected parallelopipeds, viz., the prisms of the same altitude. Therefore the prism upon the inscribed square A B C D is the half of the prism upon the circumscribed square, for they are to one another (XI. 32) as their bases. But the cylinder is less than the prism upon the circumscribed square. Therefore the prism upon the inscribed square A B C D is greater than half of the cylinder. Bisect the arcs A B, B C, C D, and D A in the points E, F, G, and H; join A E, E B, B F, F C, C G,



, D H, and H A: and each of the triang
. D H A is greater than the half of the segment of
tains it (XII. 2). Erect prisms upon each of these
angles, of the same altitude with the cylinder;
ch of these prisms is greater than half of the
ylindric segment which contains it. For, if through
he points E, F, G, and H, parallels be drawn to A B,
B C, C D, and D A, and parallelograms be completed
upon these straight lines, and parallelopipeds be
erected upon the parallelograms; the prisms upon the
triangles A E B, B F C, C G D, and D H A, are the
halves of the parallelopipeds (XII. 7, *Cor.* 2); and the cylindric segments, which are upon the circular segments, cut off by A B, B C, C D, and D A, are less than the parallelopipeds which contain them. Therefore the prisms upon the triangles A E B, B F C, C G D, and D H A, are greater than half of the cylindric segments which contain them. Therefore, if each of the arcs be bisected, and straight lines be drawn from the points of bisection to the extremities of the arcs, and upon the triangles thus made, prisms be erected of the same altitude with the cylinder, and so on; there will at length remain some cylindric segments which together are less (XII. *Lem.* 1) than the excess of the cylinder above the triple of the cone. Let these cylindric segments be those upon the circular segments A E, E B, B F, F C, C G, G D, D H, and H A. Therefore the rest of the cylinder, that is, the prism upon the polygon A E B F C G D H, is greater than the triple of the cone. But this prism is triple (XII. 7, *Cor.* 1) of the pyramid, having the same base and vertex with the cone. Therefore the pyramid upon the base A E B F C G D H is greater than the cone upon the circle A B C D; but it is also less, for the pyramid is contained within the cone, which is impossible. Therefore the cylinder is not greater than the triple of the cone.

Nor can the cylinder be less than the triple of the cone. Let it be less, if possible; therefore the cone is greater than the third part of the cylinder. In the circle A B C D inscribe a square; this square is greate than the half of the circle. Upon the square A B C D erect a pyramid having the same vertex with the cone; this pyramid is greater than th half of the cone; because, as was before demonstrated, if a square be d scribed about the circle, the square A B C D is the half of it. If up these squares there be erected parallelopipeds or prisms of the same al tude with the cone, the prism upon the inscribed square A B C D is half of that which is upon the circumscribed square; for they are to one another as their bases (XI. 32); and so are also the third parts of these solids. Therefore the pyramid upon the inscribed square A B C D is half of the pyramid upon the circumscribed square. But the latter pyramid is greater than the cone which it contains. Therefore the pyramid upon the inscribed square A B C D is greater than the half of the cone. Bisect the arcs A B, B C, C D, and D A, in the points F, F, G, and H, and join A E, E B, B F, F C, C G, G
H A. Each of the triangles A E B, B F C, C G D, and
lf of the segment of the circle which contains



each of these triangles erect pyramids having the same vertex with the cone. Each of these pyramids is greater than the half of the segment of the cone which contains it, as was before demonstrated of the prisms and segments of the cylinder: and thus bisecting each of the arcs and joining the points of bisection and their extremities by straight lines, and upon the triangles erecting pyramids having their vertices the same with that of the cone, and so on, there will at length remain some segments of the cone, which together are less (XII. *Lem.* 1) than the excess of the cone above the third part of the cylinder. Let these be the segments upon A E, E B, B F, F C, C G, G D, D H, and H A. Therefore the rest of the cone, that is, the pyramid upon the polygon A E B F C G D H, is greater than the third part of the cylinder. But this pyramid is the third part of the prism upon the same base A E B F C G D H, and of the same altitude. Therefore this prism is greater than the cylinder upon the circle A B C D; but it is also less, for it is contained within the cylinder; which is impossible. Therefore the cylinder is not less than the triple of the cone. And it has been demonstrated that neither is it greater than the triple. Therefore the cylinder is triple of the cone, or, the cone is the third part of the cylinder. Wherefore every cone, &c. Q. E. D.

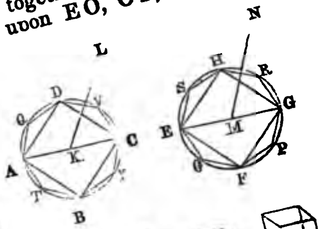
This proposition is the foundation of the rule for finding the solid content of a cone.

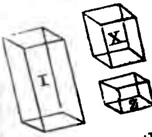
PROP. XI. THEOREM.

Cones and cylinders of the same altitude, are to one another as their bases.

Let the cones and cylinders, of which the bases are the circles A B C D and E F G H, and the axes K L and M N, and of which A C and E G are the diameters of their bases, be of the same altitude. The circle A B C D is to the circle E F G H as the cone A L is to the cone E N.

If not, the circle A B C D is to the circle E F G H as the cone A L is to some solid either less than the cone E N, or greater than it. First, let it be to a solid less than E N, viz., to the solid X; and let Z be the solid which is equal to the excess of the cone E N above the solid X; therefore the cone E N is equal to the solids X and Z together. In the circle E F G H inscribe the square EFGH; this square is greater than the half of the circle. Upon the square E F G H erect a pyramid of the same altitude with the cone; this pyramid is greater than half of the cone. For, if a square be described about the circle, and a pyramid be erected upon it, having the same vertex with the cone, the pyramid inscribed in the cone is half of the pyramid circumscribed about it, because they are to one another as their bases (XII. 6). But the cone is less than the circumscribed pyramid. Therefore the pyramid upon the square E F G H is greater than half of the cone. Bisect the arcs E F, F G, G H, and H E at the points O, P, R, and S; and join E O, O F, F P, P G, G R, R H, H S, and S E. Therefore each of the triangles E O F, F P G, G R H, and H S E is greater than half of the segment of the circle which contains it. Upon each of these triangles erect a pyramid having the same vertex with the cone; each of these pyramids is greater than the half of the segment of the cone which contains it. Bisecting each of these arcs and from the points of bisection drawing straight lines to the extremities of the arcs, and upon each of the triangles thus made erecting pyramids having the same vertex with the cone, and so on; there will at length





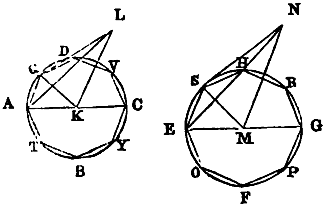
emain some segments of the cone which are together less (XII. *Lem.* 1) han the solid Z; let these be the segments upon EO, OF, FP, PG, G R, R H, H S, and S E. Therefore the remainder of the cone, viz., the pyramid upon the polygon E O F P G R H S is greater than the solid X. In the circle A B C D inscribe the polygon A T B Y C V D Q similar to the polygon E O F P G R H S, and upon it erect a pyramid having the same vertex with the cone A L. Because the square of A C is to the square of E G (XII. 1), as the polygon A T B Y C V D Q is to the polygon E O F P G R H S; and the square of A C is to the square of E G, as (XII. 2) the circle A B C D is to the circle E F G H. Therefore the circle A B C D is (V. 11) to the circle E F G H, as the polygon A T B Y C V D Q is to the polygon EOFPGRHS. But the circle A B C D is to the circle E F G H, as the cone A L is to th solid X, and the polygon A T B Y C V D Q is to the polygon EOFPGRH as (XII. 6) the pyramid whose base is the first of these polygons, a vertex L, is to the pyramid whose base is the other polygon, and vert N. Therefore the cone A L is to the solid X, as the pyramid whose ba is the polygon A T B Y C V D Q, and vertex L is to the pyramid wh base is the polygon E O F P G R H S, and vertex N. But the cone A is greater than the pyramid contained in it. Therefore the solid X greater (V. 14) than the pyramid in the cone E N: but it is also less was shown; which is absurd. Therefore the circle A B C D is not to c circle E F G H, as the cone A L is to any solid which is less than the c E N. In the same manner it may be demonstrated, that the cir E F G H is not to the circle A B C D, as the cone E N is to any solid l than the cone A L. Nor can the circle A B C D be to the circle E F G as the cone A L is to any solid greater than the cone E N. For, if it possible, let this be solid I, which is greater than the cone E N. The fore, by inversion, the circle E F G H is to the circle A B C D, as solid I is to the cone A L. But the solid I is to the cone A L, as the E N is to some solid less (V. 14) than the cone A L; because the I is greater than the cone E N. Therefore the circle E F G H is circle A B C D, as the cone E N is to a solid less than the cone A L, was shown to be impossible. Wherefore the circle A B C D is not circle E F G H, as the cone A L is to any solid greater than th E N. And it has been demonstrated, that neither is the circle to the circle E F G H, as the cone A L is to any solid less than t E N. Therefore the circle A B C D is to the circle E F G H, as A L is to the cone E N. But as the cone is to the cone, so (V. cylinder to the cylinder, because the cylinders are triple (XII. cones, each to each. Therefore the circles A B C D and E F one another as the cylinders upon them of the same altitude fore, cones and cylinders of the same altitude are to one anot bases. Q. E. D.

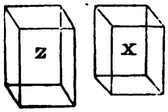
PROP. XII. THEOREM.

Similar cones and cylinders have to one another the triplicate ratio of that which the diameters of their bases have.

Let the cones and cylinders of which the circles A B C D and E F G H are the bases, A C and E G the diameter of the bases, and K L and M N the axes of the cones or cylinders, be similar. The cone whose base is the circle A B C D and vertex the point L, has to the cone whose base is the circle E F G H and vertex N, the triplicate ratio of that which A C has to E G.

For if the cone ABCDL has not to the cone E F G H N the triplicate ratio of that which AC has to E G, the cone A B C D L has the triplicate of that ratio to some solid which is less or greater than the cone E F G H N. First, let the cone have it to a less solid, viz., to the solid X. Make the same construction as in the preceding proposition, and it may be demonstrated in the same manner as in that proposition, that the pyramid of which the base is the polygon EOFPGRHS, and vertex N, is greater than the solid X. Inscribe also in the circle A B C D the polygon ATBYCVDQ similar to the polygon E O F P G R H S, upon which erect a pyramid having the same vertex with the cone; and let L A Q be one of the triangles containing the pyramid upon the polygon ATBYCVDQ whose vertex is L; and let N E S be one of the triangles containing the pyramid upon the polygon E O F P G R H S whose vertex is N. Join K Q and M S. Because the cone A B C D L is similar to the cone E F G H N. Therefore AC is (XI. *Def* 24) to E G as the axis K L to the axis M N. But A C is to E G (V. 15) as A K is to E M. Therefore A K is to E M, as K L is to M N; and alternately, A K is to K L, as E M is to M N; and the right angles A K L, E M N are equal. Therefore the sides about these equal angles are proportionals, and the triangle A K L is similar (VI. 6) to the triangle E M N. Again, because A K is to K Q as E M is to M S, and these sides are about equal angles, A K Q and E M S, each of these angles being the same part of four right angles at the centres K and M. Therefore the triangle A K Q is similar (VI. 6) to the triangle E M S. Because it has been shown that A K is to K L as E M is to M N, and that A K is equal to K Q, and E M to M S. Therefore Q K is to K L as S M is to M N; and the sides about the right angles Q K L and S M N are proportionals. Wherefore the triangle L K Q is similar to the triangle N M S. Because the triangles A K L and E M N are similar, L A is to A K as N E is to E M; and because the triangles A K Q and E M S are similar, K A is to A Q, as M E is to E S. Therefore *ex æquali* (V. 22), L A is to A Q as N E is to E S. Again, because the triangles L Q K and N S M are similar, L Q is to Q K as N S is to S M; and because the triangles

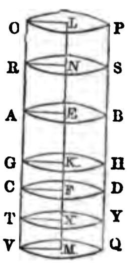




K A Q and M E S are similar, K Q is to Q A as M S is to S E. Therefore, *ex æquali* (V. 22), L Q is to Q A as N S is to S E. But it was proved that Q A is to A L as S E is to E N. Therefore, *ex æquali*, Q L is to L A as S N is to N E. Wherefore the triangles L Q A and N S E, having the sides about all their angles proportionals, are equiangular (VI. 5) and similar to one another. But the pyramid, whose base is the triangle A K Q and vertex L, is similar to the pyramid whose base is the triangle E M S, and vertex N, because their solid angles are equal (XI. B) to one another, and they are contained by the same number of similar planes. But similar triangular pyramids have to one another the triplicate (XII. 8) ratio of that which their homologous sides have. Therefore the pyramid A K Q L has to the pyramid E M S N the triplicate ratio of that which A K has to E M. In the same manner, if straight lines be drawn from the points D, V, C, Y, B, and T to K, and from the points H, R, G, P, F, and O to M, and pyramids be erected upon the triangles having the same vertices with the cones, it may be demonstrated that each pyramid in the first cone has to each in the other, taking them in the same order, the triplicate ratio of that which the side A K has to the side E N; that is, which A C has to E G. But as one antecedent is to its consequent, so are all the antecedents to all the consequents (V. 12). Therefore, as the pyramid A K Q L is to the pyramid E M S N, so is the whole pyramid whose base is the polygon D Q A T B Y C V, and vertex L, to the whole pyramid, whose base is the polygon H S E O F P G R, and vertex N. Wherefore also these pyramids have to one another the triplicate ratio of that which A C has to E G. But (*Hyp.*) the cone ABCDL has to the solid X the triplicate of that which A C has to E G. Therefore the cone A B C D L is to the solid X, as the pyramid upon the polygon D Q A T B Y C V, with vertex L, is to the pyramid upon the polygon H S E O F P G R, with vertex N. But the cone A B C D L is greater than the pyramid contained in it. Therefore the solid X is greater (V. 14) than the pyramid upon the polygon H S E O F P G R, with vertex N; but it is also less; which is impossible. Therefore the cone A B C D L has not to any solid less than the cone E F G H N the triplicate ratio of that which A C has to E G. In the same manner it may be demonstrated that neither has the cone E F G H N to any solid less than the cone A B C D L the triplicate ratio of that which E G has to A C. Nor can the cone A B C D L have to any solid greater than the cone E F G H N the triplicate ratio of that which A C has to E G. For, if it be possible, let this cone have it to a greater, viz., to the solid Z. Therefore, inversely, the solid Z has to the cone A B C D L the triplicate ratio of that which E G has to A C. But the solid Z is to the cone ABCDL as the cone EFGHN is to some solid, which must be less (V. 14) than the cone A B C D L, because the solid Z is greater than the cone EFGHN. Therefore the cone E F G H N has to a solid which is less than the cone A B C D L the triplicate ratio of that which E G has to A C, which was demonstrated to be impossible. Therefore the cone A B C D L has not to any solid greater than the cone E F G H N the triplicate ratio of that which A C has to E G; and it was demonstrated, that it could not have that ratio to any solid less than the cone E F G H N. Therefore the cone ABCDL has to the cone EFGHN the triplicate ratio of that which A C has to E G. But the cone is to the cone as (V. 15) the cylinder is to the cylinder; for every cone is the third (XII. 10) part of the cylinder upon the same base, and of the same altitude. Therefore also the cylinder has to the cylinder, the triplicate ratio of that which A C has to E G. Wherefore, similar cones, &c. Q. E. D.

PROP. XIII. THEOREM.

If a cylinder be cut by a plane parallel to its opposite planes, or bases, it divides the cylinder into two cylinders, which are to one another as their axes.



Let the cylinder A D be cut by the plane G H parallel to the opposite planes A B and C D, and meeting the axis E F in the point K; let the line G H be the common section of the plane G H and the surface of the cylinder A D. Let A F be the generating rectangle of the cylinder, and E F its fixed side; and let G K be the common section of the plane G H, and the plane A F. Because the parallel planes A B and G H are cut by the plane A K, A E and K G, their common sections with it, are parallel (XI. 16). Wherefore A K is a rectangle, and G K is equal to E A the straight line from the centre of the circle A B. For the same reason, each of the straight lines drawn from the point K to the line G H may be proved to be equal to those which are drawn from the centre of the circle A B to its circumference. Wherefore they are all equal to one another. Therefore the line G H is the circumference of a circle (I. *Def.* 15) of which the centre is the point K; and the plane G H divides the cylinder A D into the two cylinders A H and G D; for they are the same which would be generated by the rectangles A H and G F. It is required to show that the cylinder A H is to the cylinder H C, as the axis E K is to the axis K F.

Produce the axis E F both ways: and take any number of straight lines E N and N L, each equal to E K; and any number F X and X M, each equal to F K; and let planes parallel to A B or C D, pass through the points L, N, X, and M. The common sections of these planes with the cylinder produced, are circles, the centres of which are the points L, N, X, and M, as was proved of the plane G H; and these planes cut off the cylinders P R, R B, D T, and T Q. Because the axes L N, N E, and E K, are all equal. Therefore the cylinders P R, R B, and B G, are (XII. 11) to one another as their bases. But their bases are equal. Therefore the cylinders P R, R B, and B G, are all equal. Because the axes L N, N E, and E K, are equal to one another, as also the cylinders P R, R B, and B G, and there are as many axes as cylinders. Therefore, whatever multiple the axis K L is of the axis K E, the same multiple is the cylinder P G of the cylinder G B. For the same reason, whatever multiple the axis M K is of the axis K F, the same multiple is the cylinder Q G of the cylinder G D. But if the axis K L be equal to the axis K M, the cylinder P G is equal to the cylinder G Q; if the axis K L be greater than the axis K M, the cylinder P G is greater than the cylinder G Q; and if less, less. Therefore, there are four magnitudes, viz., the axes E K and K F, and the cylinders B G and G D; and of the axis E K, and cylinder B G, there have been taken any equimultiples whatever, viz., the axis K L and cylinder P G; and of the axis K F, and cylinder G D, any equimultiples whatever, viz., the axis K M and cylinder G Q; and it has been demonstrated, that if the axis K L be greater than the axis K M, the cylinder PG is greater than the cylinder GQ; if equal, equal; and if less, less. Therefore (v. *Def.* 5), the axis E K is to the axis K F, as the cylinder B G is to the cylinder G D. Wherefore, if a cylinder, &c. Q. E. D.

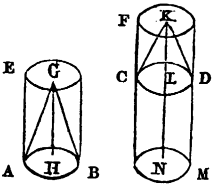
Corollary.—**Every section of a cylinder by a plane parallel to its base is a circle and, every section by a plane through its axis is a rectangle.**

PROP. XIV. THEOREM.

Cones and Cylinders upon equal bases are to one another as their altitudes.

Let the cylinders E B and F D be upon the equal bases A B and C D. The cylinder E B is to the cylinder F D as the axis G H is to the axis K L.

Produce the axis K L to the point N, make L N equal to the axis G H, and let C M be a cylinder of which the base is C D, and axis L N.



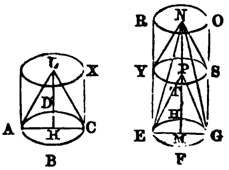
Because the cylinders E B and C M have the same altitude, they are to one another as their bases (XII. 11). But their bases are equal. Therefore, also, the cylinders E B and C M are equal. Because the cylinder F M is cut by the plane CD parallel to its opposite planes, the cylinder C M is to the cylinder FD, as (XII. 13) the axis L N is to the axis K L. But the cylinder C M is equal to the cylinder E B, and the axis L N to the axis G H. Therefore, the cylinder E B is to the cylinder F D as the axis G H is to the axis K L; and the cylinder E B is to the cylinder F D as (V. 15) the cone A B G is to the cone C D K, because the cylinders are triple (XII. 10) of the cones. Therefore, the axis G H is to the axis K L, as the cone A B G is to the cone C D K, and as the cylinder E B is to the cylinder F D. Wherefore cones, &c. Q. E. D.

PROP. XV. THEOREM.

The bases and altitudes of equal cones and cylinders are reciprocally proportional; and if the bases and altitudes of cones and cylinders be reciprocally proportional, they are equal to one another.

Let the circles A B C D and E F G H, the diameters of which are A C and E G, be the bases, and K L and M N the axes, or the altitudes, of the equal cones A L C and E N G, and of the equal cylinders A X and E O. The bases and altitudes of the cylinders A X and E O are reciprocally proportional; that is, the base A B C D is to the base E F G H as the altitude M N is to the altitude K L.

First, when the altitudes M N and K L are equal. Because the cylinders A X and E O are also equal, and cones and cylinders of the same altitude are to one another as their bases (XII. 11). Therefore the base A B C D is equal (A 5) to the base E F G H. Wherefore the base A B C D is to the base E F G H as the altitude M N is to the altitude K L. Next, when the altitudes K L and M N are unequal, and M N the greater of the two. From M N take M P equal to K L, and through *the point P let* the cylinder E O be cut by the plane T Y S, parallel *to the opposite* planes of the circles F H and R O. Because the common section of the plane T Y S and the cylinder E O is a circle (XII. 13 *Cor.*) E S is a cylinder, of which the base is the circle E F G H, and altitude M P. Because the cylinder A X is equal to the cylinder E O. The cylinder A X is to the cylinder E S as (V. 7) the cylinder E O is to the cylinder E S. But the cylinder A X is to the cylinder E S (XII. 11) as the base A B C D is to the base E F G H; for the cylinders A X and E S are of the same altitude; and the cylinder E O is to the cylinder E S (XII. 13) as the altitude M N is to the altitude M P, because the cylinder E O is cut by the plane T Y S parallel to its opposite planes. Therefore the base A B C D is to the base E F G H as the altitude M N is to the altitude M P. But M P is equal to K L. Therefore the base A B C D is to the base E F G H as the altitude M N is to the altitude K L; that is, the bases and altitudes of the equal cylinders A X and E O are reciprocally proportional.



Next, let the bases and altitudes of the cylinders A X and E O be reciprocally proportional; viz., the base A B C D to the base E F G H as the altitude M N to the altitude K L. The cylinder A X is equal to the cylinder E O.

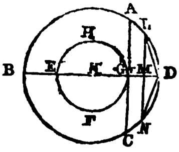
First, let the base A B C D be equal to the base E F G H. Because the base A B C D is to the base E F G H as the altitude M N is to the altitude K L; M N is equal (V. A) to K L. Therefore the cylinder A X is equal (XII. 11) to the cylinder E O.

But let the bases A B C D and E F G H be unequal, and let A B C D be the greater. Because A B C D is to the base E F G H as the altitude M N is to the altitude K L. Therefore M N is greater (V. A) than K L. The same construction being made as before, because the base A B C D is to the base E F G H as the altitude M N is to the altitude K L; and, because the altitude K L is equal to the altitude M P. Therefore the base A B C D is (XII. 11) to the base E F G H as the cylinder A X is to the cylinder E S; and the altitude M N is to the altitude M P or K L, as the cylinder E O is to the cylinder E S. Therefore the cylinder A X is to the cylinder E S as the cylinder E O is to the cylinder E S. Therefore the cylinder A X is equal to the cylinder E O: and the same reasoning holds in cones. Wherefore the bases, &c. Q. E. D.

PROP. XVI. PROBLEM.

In the greater of two concentric circles to inscribe a polygon of an even number of equal sides, that shall not meet the less circle.

Let A B C D and E F G H be two given circles having the same centre K. It is required to inscribe in the greater circle A B C D, a polygon of an even number of equal sides that shall not meet the less circle E F G H.



Through the centre K draw the straight line B D, and from the point G, where it meets the circumference of the less circle, draw A C touching (III. 17) the circle E F G H. If the circumference B A D be bisected, and the half of it be again bisected, and so on, there will at length remain an arc less

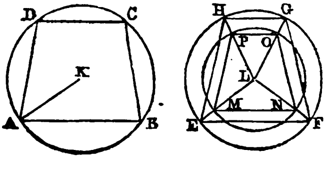
(XII. *Lemma* 1) than A D: let this arc be L D. From the point L draw L N parallel to B D, and join L D and D N. Because L D is equal to D N (I. 4) L N is parallel to A C, and A C touches the circle E F G H. Therefore L N does not meet the circle E F G H; and much less do the straight lines L D and D N, meet the circle E F G H. Therefore, if straight lines equal to L D be applied in the circle A B C D from the point L all round to N, there will be inscribed in the circle a polygon of an even number of equal sides not meeting the less circle. Q. E. F.

LEMMA II.

If two trapeziums A B C D and E F G H be inscribed in the circles, the centres of which are the points K and L; and if the sides A B and D C be parallel, as also E F and H G; and the other four sides A D, B C, E H, and F G, be all equal to one another; but the side A B greater than the side E F, and the side D C greater than the side H G; the radius K A of the circle in which the greater sides are, is greater than the radius L E of the other circle.

If it be possible, let K A be not greater than L E; then K A must be either equal to L E, or less than it. First, let K A be equal to L E. Because in two equal circles, the two straight lines A D and B C, in the one, are equal to two straight lines E H and F G in the other, the arcs A D and B C are equal (III. 28) to the arcs E H and F G. But the straight lines A B and D C are respectively greater than the straight lines E F and G H. Therefore the arcs A B and D C are greater than the arcs E F and H G. Wherefore the circumference A B C D is greater than the circumference E F G H; but it is also equal to it; which is impossible. Therefore the radius K A is not equal to the radius L E.

But let K A be less than L E; make L M equal to K A, and from the centre L, and distance L M, describe the circle M N O P, meeting the straight lines L E, L F, L G, and L H, in M, N, O, and P. Join M N, N O, O P, and P M; these straight lines are respectively parallel (VI. 2) to and less than the straight lines E F, F G, G H, and H E. Because E H is greater than M P, A D is greater than M P. But the circles A B C D, and M N O P are equal. Therefore the arc A D is greater than the arc M P. For the same reason, the arc B C is greater than the arc N O. Because A B is greater than E F, and E F is greater than M N; much more is A B greater than M N. Therefore the arc A B is greater than the arc M N. For the same reason, the arc D C is greater than the arc P O. Therefore the circumference A B C D is greater than the circumference M N O P; but they are likewise equal, which is impossible. Therefore K A is not less than L E; nor is K A equal to K E. Therefore the radius K A is greater than the radius L E. Wherefore if two, &c., Q. E. D.



Corollary.—If there be an isosceles triangle, the sides of which are equal to A D and B C, but the base less than A B, the greater of the two sides A B and C D, *it may be* shown also that K A is greater than the radius of the circle described *about the triangle.*

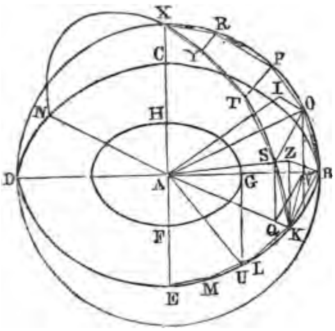
PROP. XVII. PROBLEM.

In the greater of two spheres which have the same centre, to inscribe a polyhedron the superficies or faces of which shall not meet the less.

Let there be two spheres having the same centre A. It is required to inscribe in the greater sphere a solid polyhedron, the superficies of which shall not meet the less.

Let the spheres be cut by a plane passing through the centre; the common sections of it with the spheres are circles; for the sphere is described by the revolution of a semicircle about the diameter remaining fixed. Therefore, in whatever position the semicircle is supposed to be, the common section of the plane in which it is, with the superficies of the sphere, is the circumference of a circle; and this section is a great circle of the sphere, because the diameter of the sphere, which is also the diameter of the circle, is greater (III. 15) than any chord in the circle or the sphere. Let the circle made by the section of the plane with the greater sphere be, B C D E, and that with the less, F G H. Draw the two diameters B D and C E at right angles to one another; and in B C D E, the greater of the two circles, inscribe (XII. 16) a polygon of an even number of equal sides not meeting the less circle F G H. Let the sides of this polygon in B E, the fourth part of the circle, be B K, K L, L M, and M E. Join K A, and produce it to N. From A draw (XI. 12) A X at right angles to the plane of the circle B C D E, and let it meet the superficies of the sphere in the point X. Let planes pass through the straight line A X, and each of the straight lines B D and K N; their sections with the superficies of the sphere are great circles. Let B X D and K X N be their semicircles, and B D and K N their diameters. Because X A is at right angles to the circle B C D E, every plane which passes through X A is at right (XI. 18) angles to the circle B C D E. Therefore the semicircles B X D and K X N are at right angles to B C D E. Because the semicircles B E D, B X D, and K X N upon the equal diameters B D and K N, are equal, their halves B E, B X, and K X, are equal. Therefore as many sides of the polygon as are in B E, so many are there in B X and K X, equal to the sides B K, K L, L M, and M E. Let these polygons be described, and let their sides be B O, O P, P R, and R X; K S, S T, T Y, and Y X. Join O S, P T, and R Y; and from the points O and S, draw O V and S Q perpendiculars to A B and A K.

Because the plane B O X D is at right angles to the plane B C D E, and in B O X D, O V is drawn perpendicular to A B, the common section of the planes, therefore O V is perpendicular (XI. *Def.* 4) to the plane B C D E. For the same reason S Q is perpendicular to the same plane, the plane K S X being at right angles to the plane B C D E. Join V Q. Because in the equal semicircles B X D and K X N, the arcs B O and K S are equal, and O V and S Q are perpendiculars to their diameters. Therefore (I. 26) O V is equal to S Q, and B V to K Q. But the whole B A is equal to the whole K A. Therefore the remainder V A is equal to the remainder Q A; and B V is to V A as K Q is to Q A. Wherefore V Q is parallel (VI. 2) to B K. Because O V and S Q are each of them at right angles to the circle B C D E, O V is parallel (XI. 6) to S Q; and it has been proved that it is also equal to it. Therefore Q V and S O are equal (I. 33) and parallel. Because Q V is parallel to S O, and also to K B; O S is parallel (xi. 9) to B K. Therefore B O and K S, which join them, are in the same plane, and the quadrilateral figure K B O S is in that plane. If P B and T K be joined and perpendiculars be drawn from the points P and T to the straight lines A B and A K, it may be proved, in the same manner, that T P is parallel to K B. Therefore (XI. 9) T P being parallel to S O, the quadrilateral figure S O P T is in one plane. For the same reason, the quadrilateral T P R Y is in one plane, and the figure Y R X (XI. 2) is also in one plane. Therefore, if from the points O, S, P, T, R, and Y, straight lines be drawn to the point A, a polyhedrôn will be formed between the arcs B X and K X, composed of pyramids, whose bases are the quadrilaterals K B O S, S O P T, T P R Y, and the triangle Y R P, and whose common vertex is the point A. If the same construction be made upon each of the sides K L, L M and M E, as upon B K, and the same be done also in the other three quadrants, and in the other hemisphere; a solid polyhedron will be inscribed in the sphere, composed of pyramids, whose bases are the quadrilaterals K B O S, &c., and triangle Y R X, and those formed in the same manner in the rest of the sphere, and common vertex the point A.



Also the superficies of this polyhedron does not meet the less sphere of which the circle F G H is a section. For, from the point A draw (XI. 11) A Z perpendicular to the plane of the quadrilateral K B O S, meeting it in Z, and join B Z and Z K. Because A Z is perpendicular to the plane K B O S, it makes right angles with every straight line meeting in that plane. Therefore A Z is perpendicular to B Z and K Z; and it may be shown, as in Prop. XIV, Book III, that the straight line B Z is equal to the straight line Z K. In the same manner it may be shown that the straight lines drawn from the point Z to the points O and S, are equal to B Z or *Z* K. Therefore the circle described from the centre Z, and distance Z B, will pass through the points K, O and Z, and K B O S will be a quadrilateral inscribed in the circle. Because K B is greater than Q V, and Q V equal to S O. Therefore K B is greater than S O. But K B is equal to each of the straight lines B O and K S. Therefore each of the arcs cut off by K B, B O and K S, is greater than that cut off by O S; and these three arcs together with a fourth equal to each, are greater than the same three together with that cut off by O S; that is, than the whole circumference of the circle. Therefore the arc subtended by K B is greater than the fourth part of the whole circumference of the circle about K B O S, and the angle B Z K at the centre is greater than a right angle. From the point G draw G U at right angles to A G, and join A U. If then the circumference B E be bisected, and its half again bisected, and so on, there will at length be left an arc less than the arc subtended by a straight line equal to G U, incribed in the circle B C D E; let this be the circum*ference* K B. Therefore the straight line K B is less than G U. Because *the angle B Z K* is obtuse, B K is greater than B Z; in the triangles *A G U* and A B Z, it may be shown, as in Prop. XV, Book III, that A Z is greater than A G. But A Z is perpendicular to the plane K B O S, and is, therefore, the shortest of all the straight lines that can be drawn from A, the centre of the sphere, to that plane. Therefore the plane K B O S does not meet the less sphere.

Also the other planes between the quadrants B X and K X do not meet the less sphere. From the point A draw A I perpendicular to the plane of the quadrilateral S O P T, and join I O. In the same manner, as it was proved of the plane K B O S and the point Z, it may be shown that the point I is the centre of a circle described about S O P T; and that O S is greater than P T. But P T was shown to be parallel to O S; and because the two trapeziums K B O S and S O P T inscribed in circles have their sides B K and O S parallel, as also O S and P T; and their other sides B O, K S, O P, and S T all equal to one another, but B K is greater than O S, and O S greater than P T. Therefore the straight line Z B is greater (XII. *Lemma* 2) than I O. Join A O. In the triangles A B Z and A I O, it may be shown, as above, that A Z is less than A I. But it was proved that A Z is greater than A G; much more then is A I greater than A G. Therefore the plane S O P T does not meet the less sphere. In the same manner it may be proved, that the plane T P R Y does not meet the same sphere, as also the triangle Y R X (XII. *Lem.* 2, *Cor.*) In the same manner it may be demonstrated, that all the planes, or faces the polyhedron, do not meet the less sphere. Therefore in the greater of two spheres, which have the same centre, a polyhedron is inscribed, the superficies of which does not meet the lesser sphere. Q. E. F.

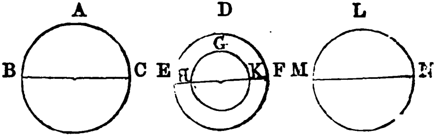
COROLLARY.—If in the less sphere a similar polyhedron be inscribed by joining the points in which the radii of the sphere drawn to the angles of the polyhedron in the greater sphere meet the superficies of the less, the polyhedron in the sphere B C D E shall have to this polyhedron, the triplicate ratio of that which the diameter of the sphere B C D E has to the diameter of the less sphere. For if these two solids be divided into the same number of pyramids, and in the same order, the pyramids shall be similar to one another, each to each : because they have the solid angles at their common vertex, the centre of the sphere, the same in each pyramid, and their other solid angles at the bases, equal to one another, each to each (XI .*B*), because they are contained by three plane angles, each equal to each; and the pyramids are contained by the same number of similar planes; and are therefore similar (XI. *Def.* 11) to one another, each to each. But similar pyramids have to one another the triplicate (XII. 8, *Cor.*) ratio of their homologous sides. Therefore the pyramid of which the base is the quadrilateral K B O S, and vertex A, has to the pyramid in the inner sphere of the same order, the triplicate ratio of their homologous sides, that is, of that ratio which the radius A B, the greater sphere, has to the radius of the less sphere. In like manner, each pyramid in the greater sphere has to each of the same order in the less, the triplicate ratio of that which A B has to the radius of the less sphere. But as one antecedent is to its consequent, so are all the antecedents to all the consequents. Wherefore the whole polyhedron in the greater sphere has to the whole polyhedron in the less, the triplicate ratio of that which A B has to the radius of the less; that is, which the diameter B D of the greater has to the diameter of the less sphere.

Corollary 2.—Every face of a polyhedron inscribed in a sphere is inscribable in a circle.

PROP. XVIII. THEOREM.

Spheres have to one another the triplicate ratio of that which their diameters have.

Let A B C and D E F be two spheres, of which the diameters are B C and E F. The sphere A B C has to the sphere D E F the triplicate ratio of that which B C has to E F.



For, if not, the sphere A B C must have to a sphere either less or greater than D E F, the triplicate ratio of that which B C has to E F. First, let it have that ratio to a less, viz., to the sphere G H K; and let the sphere D E F have the same centre with G H K. In the greater sphere D E F inscribe (XII. 17) a polyhedron, the superficies of which does not meet the lesser sphere G H K; and in the sphere A B C inscribe another similar to that in the sphere D E F. Therefore the polyhedron in the sphere A B C has to the polyhedron in the sphere D E F the triplicate ratio (XII. *Cor.* 17) of that which B C has to E F. But the sphere A B C has to the sphere G H K, the triplicate ratio of that which B C has to E F. Therefore, as the sphere A B C is to the sphere G H K, so is the polyhedron in the sphere A B C to the polyhedron in the sphere D E F. But the sphere A B C is greater than the inscribed polyhedron. Therefore (V. 14), also the sphere G H K is greater than the polyhedron in the sphere D E F; but it is also less, because it is contained within it, which is impossible. Therefore, the sphere A B C has not to any sphere less than D E F, the triplicate ratio of that which B C has to E F. In the same manner it may be demonstrated, that the sphere D E F has not to any sphere less than A B C, the triplicate ratio of that which E F has to B C. Nor can the sphere A B C have to any sphere greater than D E F, the triplicate ratio of that which B C has to E F; for, if it can, let it have that ratio to a greater sphere L M N. Therefore, by inversion, the sphere L M N has to the sphere A B C, the triplicate ratio of that which the diameter E F has to the diameter B C. But the sphere L M N is to A B C as the sphere D E F is to some sphere, which must be less (V. 14) than the sphere A B C, because the sphere L M N is greater than the sphere D E F. Therefore, the sphere D E F has to a sphere less than A B C the triplicate ratio of that which E F has to B C; which was shown to be impossible. Therefore, the sphere A B C has not to any sphere greater than D E F, the triplicate ratio of that which B C has to E F; and it was demonstrated, that neither has it that ratio to any sphere less than D E F. Therefore, the sphere A B C has to the sphere D E F, the triplicate ratio of that which B C has to E F. Wherefore, spheres have to one another, &c.

Every sphere is two-thirds of its circumscribed cylinder.—*Archimedes.*

FINIS.

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